Introduction to Variational Methods in Imaging

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Outline

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- Introduction to calculus of variations
- Numerical methods for variational minimization
- Applications of variational methods in imaging
- Conclusion
- Acknowledgments
Common image processing tasks

- De-noising: Remove noise from an image
- Segmentation: Partition the image into object and background
- Optic flow: Estimate the apparent motion between two images
- Registration: Transform the source image to match the template image
**Image processing tasks as function estimation**

- **De-noising:** Find a smooth approximation to the noisy image in the space of images.
- **Segmentation:** Find a smooth closed curve between object and background.
- **Optic flow:** Compute a smooth displacement field between two images.
- **Registration:** Estimate a smooth and realistic deformation field that matches the corresponding points in template and source images.
Motivation (2)

Image processing tasks as function estimation

General variational framework

- Goal: To determine an unknown function $u(x)$ satisfying given constraints

- The constraints are formulated in the form of an energy functional as follows:

$$E[u] = \int_{\Omega} \left[ D(u) + S(u) + T(u) \right] dx$$

- Using calculus of variations, determine the unknown function as the argument that minimizes the above energy:

$$u^* = \underset{u \in U}{\text{argmin}} \, E[u]$$
Variational image de-noising

Consider the variational de-noising of some noisy image $I_0$, i.e., find the minimizer $I_\alpha$ of:

$$E[I] = \int_\Omega (I - I_0)^2 + \alpha \Psi(\nabla u)^2 dx$$

- In the above the, first term (data term, similarity term, fidelity term) encourages similarity to the original noisy image.
- Second term (smoothness term, regularizer, penalizer) encodes the smoothness constraint!
- $\alpha > 0$ is the regularization parameter (smoothness weight)
Variational image de-noising...

Now, we seek a minimizer of the functional $E[I]$:

$$I_\alpha = \arg\min_{I \in \mathcal{I}} E[I] = \arg\min_{I \in \mathcal{I}} \int_{\Omega} (I - I_0)^2 + \alpha \Psi(||\nabla u||^2) \, dx$$

$I_\alpha$ then corresponds to the non-noisy or smoothed image.

Calculus of variations gives the minimum of $E[I]$ as the solution of the Euler-Lagrange equation:

$$I - I_0 - \alpha \, \text{div} (\Psi' (||\nabla u||^2) \nabla u) = 0$$

$$\Rightarrow \frac{I - I_0}{\alpha} - \text{div} (\Psi' (||\nabla u||^2) \nabla u) = 0$$

This solved using gradient descent as:

$$\frac{\partial I}{\partial t} = \text{div} (\Psi' (||\nabla u||^2) \nabla u) - \frac{I - I_0}{\alpha}$$
Variational image de-noising . . .

Common choices for the penalizer:

- **Quadratic:** $\Psi(s^2) = s^2$
- **Perona-Malik** [Perona et al., 1990]: $\Psi(s^2) = \lambda^2 \log(1 + \frac{s^2}{\lambda^2})$
- **Charbonnier** [Charbonnier et al., 1994]: $\Psi(s^2) = 2\lambda^2 \sqrt{1 + \frac{s^2}{\lambda^2}} - 2\lambda^2$
- **Better (edge-preserving) smoothing by the Perona-Malik regularizer!**
Variational image de-noising . . .

- Common choices for the penalizer:
  - Quadratic: $\Psi = \lambda^2 s^2$
  - Perona-Malik [Perona et al., 1990]: $\Psi = \lambda^2 s^2 \log 1 + \lambda^2$
  - Charbonnier [Charbonnier et al., 1994]: $\Psi = 2\lambda^2$

Better (edge-preserving) smoothing by the Perona-Malik regularizer!

What’s next?
- Other applications in imaging: Optic flow, Segmentation, Registration
- General framework for the numerical solution of variational minimization
- ...But first formal introduction to the calculus of variations
Functionals

- A **functional** is a correspondence that assigns a real number to each function belonging to a class.

- The expression

\[
E[y] = \int_{\Omega} F[x, y(x), \nabla y(x)] \, dx
\]

defines a functional \( E[y] \), where \( y(x) \in \mathcal{D}_1(\Omega) \).

- For example, we have already seen the functional for de-noising an image \( I_0 \), where:

\[
F[I, \nabla I] = (I - I_0)^2 + \alpha \Psi(||\nabla I||^2)
\]
Introduction to calculus of variations (2)

Minimization of functionals

Consider an increment \( h(x) \) in the “independent” variable \( y(x) \), we can then calculate the increment in the function \( E[y] \) as:

\[
\triangle E[y] = E[y + h] - E[y] = \int_{\Omega} [F(x, y + h, \nabla y + \nabla h) - F(x, y, \nabla y)] \, dx
\]

Using Taylor’s theorem to expand the integrand, we obtain

\[
\triangle E[y] = \int_{\Omega} [F_y(x, y, \nabla h) h - F_{\nabla y}(x, y, \nabla y)^T \nabla h] \, dx + O(h)
\]

Ignoring the higher order terms and simplifying the notation we get the first variation of the above functional as:

\[
\delta E[y] = \int_{\Omega} [F_y h - F_{\nabla y}^T \nabla h] \, dx
\]
Minimization of functionals ...

The necessary integral condition for the extremum is

$$\delta E[y] = \int_{\Omega} [F_y h - F_{\nabla y}^T \nabla h] \, d\mathbf{x} = 0 \quad \forall \, h \in D_1(\Omega)$$

We then obtain the corresponding Euler-Lagrange equation as:

$$F_y - \text{div}(F_{\nabla y}) = 0$$

This also gives rise to the so-called natural (Neumann) boundary conditions:

$$\mathbf{n}^T F_{\nabla y} = 0$$

(see pages 152 – 154 [Gelfand et al., 1963])
**Method 1: Gradient descent**

- Set up a gradient descent evolution and discretize the resulting *parabolic* equation using Finite Differences (FD):

\[
\frac{\partial y}{\partial t} = -(F_y - \text{div}(F \nabla y))
\]

Using a *time explicit* scheme, in 2D for the above we have:

\[
y^{(k+1)}_{i,j} - y^{(k)}_{i,j} \quad \frac{\Delta t}{\triangle t} = -(F_y - \text{div}(F \nabla y))^{(k)}_{i,j}
\]

- results in a set of \( N \) (number of grid points) *linear* equations in general
- the discretization of the grid is usually *UNIFORM*, i.e. for \( i = \{1, 2, \ldots L\}, j = \{1, 2, \ldots W\} \) we have \( x_{i+1,j} - x_{i,j} = \Delta L \), \( x_{i,j+1} - x_{i,j} = \Delta W \)
Numerical methods for variational minimization (2)

Method 2: Time-lagged non-linearity

- Using Finite Differences (FD) discretize the elliptic equation:

\[(F_y - \text{div}(F_{\nabla y}))_{i,j} = 0\]
\[\equiv a_{i,j}(y)y - b_{i,j}(y) = 0\]

where \(a_{i,j}(\cdot), b_{i,j}(\cdot)\) can be non-linear \(y = \{y_{i,j}\}, a = \{a_{i,j}\}\)

- results in a set of \(N\) (number of grid points) non-linear equations

- the discretization of the grid is usually UNIFORM, i.e. for \(i = \{1, 2, \ldots L\}, j = \{1, 2, \ldots W\}\) we have \(x_{i+1,j} - x_{i,j} = \Delta L, x_{i,j+1} - x_{i,j} = \Delta W\)

- Solve the above set of non-linear equations as a series of set of linear equations

- To obtain the current estimate \(y^{k+1}\) approximate the non-linear terms using the previous estimate \(y^k\) and obtain a linear system of the following form:

\[a_{i,j}(y_{i,j}^k)y_{i,j}^{k+1} = b_{i,j}(y_{i,j}^k)\]
Method 3: Finite Element Method

- Solve integral extremum condition using the Finite Element Method (FEM)

\[ \delta E[y] = \int_{\Omega} [F_y h - F_{\nabla y}^T \nabla h] \, dx = 0 \quad \forall \, h \in \mathcal{D}_1(\Omega) \]

approximate \( y \) using nodal basis functions:

\[ y(x) \approx \sum_{n=1}^{N} y(P_n) \phi_n(x) \quad \forall \, x \in \mathbb{R}^{L \times W} \]

- Setting \( h = \phi_i, \, i = \{1, 2, \ldots, N\} \), we get \( N \) linear equations
- the discretization of the grid is usually \text{NON-UNIFORM} adapted to the problem domain and \( N \ll L \times W \)
Applications of variational methods in imaging (1)

Segmentation

- Chan-Vese variational segmentation model [Chan et al., 2001]:

\[ E[\Phi] = \int_{\Omega} \left( \frac{H(\Phi)(l - \mu_1)^2 + (1 - H(\Phi))(l - \mu_2)^2}{\text{data}} + \nu \left\| \nabla H(\Phi) \right\| \text{smoothing} \right) \, dx \]

- \( \Phi \) is the level-set function, the segmentation boundary \( \partial\Omega = \{x \mid \Phi(x) = 0\} \)

- Separate the image domain into two regions of maximally distinct average intensities \((\mu_1, \mu_2)\) while keeping the boundary length \(\left\| \nabla H(\Phi) \right\|\) small
Applications of variational methods in imaging (2)

Segmentation . . .

- The gradient descent evolution equation is given by:

\[
\frac{\partial \Phi}{\partial t} = \delta(\Phi) \left( (I - \mu_2)^2 - (I - \mu_1)^2 + \nu \text{div} \left( \frac{\nabla \Phi}{||\nabla \Phi||} \right) \right)
\]

- Region-based segmentation, NOT sensitive to initialization and noise
- Level-set representation easily handles topological changes
Optic flow

- Optic flow refers to the apparent motion of the scene between two consecutive image frames.
- The goal is to compute the displacement field that maps the pixels in the first image to their new locations in the second image.
- We assume brightness constancy and small displacements (linearization) for each pixel:

\[ |I(x, y, t) - I(x + u, y + v, t + 1)| \approx |l_x u + l_y v + l_t| = 0 \]

- 1 equation 2 unknowns \((u, v)\) at each pixel!
Applications of variational methods in imaging (4)

Variational optic flow method

\[ E[u, v] = \int_{\Omega} \left( (l_x u + l_y v + l_t)^2 + \alpha \Psi(||\nabla u||^2 + ||\nabla v||^2) \right) dx \]

- **Data term** penalizes deviations from brightness constancy (linearized)
- **Smoothness term** penalizes deviations from a smooth flow field.
  - \( \Psi(s^2) = s^2 \): **homogeneous** flow field [Horn and Schunck, 1981]
  - \( \Psi(s^2) = \sqrt{s^2 + \epsilon^2} \): **piecewise smooth** flow field [Schnörr, 1994]
- **Filling-in-effect**: in homogeneous regions WITHOUT edges, \( l_x = l_y \approx 0 \)
  \[ \Longrightarrow \quad l_x u + l_y v \approx l_t, \text{ i.e NO contribution of data term w.r.t } (u, v), \text{ smoothing term propagates or “fills in” information from neighboring regions} \]

(Images taken from [Bruhn, 2009])

Edge information

Filling-in

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Applications of variational methods in imaging (5)

Variational optic flow method . . .

(Images taken from [Bruhn, 2009])

\[ l(x, y, t) \]
\[ l(x + u, y + v, t + 1) \]
\[ \Psi(s^2) = s^2 \] (Quadratic)
\[ \Psi(s^2) = \sqrt{s^2 + \epsilon^2} \] (Linear)

\[ l_x u + l_x l_y v + l_x l_t - \alpha \text{div}(D \nabla u) = 0 \]
\[ l_x l_y u + l_y^2 v + l_y l_t - \alpha \text{div}(D \nabla v) = 0 \]

where \( D = \Psi'(||\nabla u||^2 + ||\nabla v||^2) \)

- 2 non-linear Euler-Lagrange equations
- Solved using the time-lagged non-linearity method

- The linear penalizer prevents smoothing over flow edges and hence preserves discontinuities in the flow field
Image registration

- Similar to optic flow, estimate a realistic displacement (deformation) field mapping corresponding pixels in the template image to the source image.
- The source image is warped (based on the deformation field) using interpolation to obtain the registered image.
- Challenges: Large displacements, images from different modalities, need realistic regularizers (smoothing terms).
Applications of variational methods in imaging (7)

Image registration . . .

\[ E[u, v] = - \int_{[0,255]^2} p[l_1, \hat{l}_2] \log \frac{p[l_1, \hat{l}_2]}{p[l_1] p[\hat{l}_2]} \, da \, db + \alpha \int_{\Omega} \left( \|\nabla u\|^2 + \|\nabla v\|^2 \right) \, dx \]

\[ p \equiv p(a, b) \]

- **Mutual information** is used to compute the similarity between the two different image modalities (data term NOT linearized)

- Solved using lagged non-linearity with image **warping** at each step, i.e.

\[ \hat{l}_2^{(k)} = l_2(x + u^{(k)}, y + v^{(k)}) \]
Image registration . . .

\[ E[u, v] = \int_{\Omega} (l_1 - \hat{l}_2)^2 + \mu (||\nabla u||^2 + ||\nabla v||^2) + (\lambda + \mu)(u_x + v_y)^2 \, dx \, dy \]

where \( \hat{l}_2 = l_2(x + u, y + v) \)

- Additional smoothing term based on elasticity theory, i.e., \( \text{div}\left(\frac{u}{v}\right) = 0 \) (no sources or sinks)
Conclusions

- Many image processing tasks can be posed as variational problems, which can then be solved in a common energy minimization framework.
- The variational formulations in general can be easily extended (modified) to incorporate additional (different) set of constraints.
- Efficient numerical techniques (both FD-based, FEM-based) exist that can provide a fast and an accurate solution to the variational minimization.
- Other applications in imaging and vision include stereo, structure from motion, shape estimation.
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