Fast FEM-based Non-Rigid Registration

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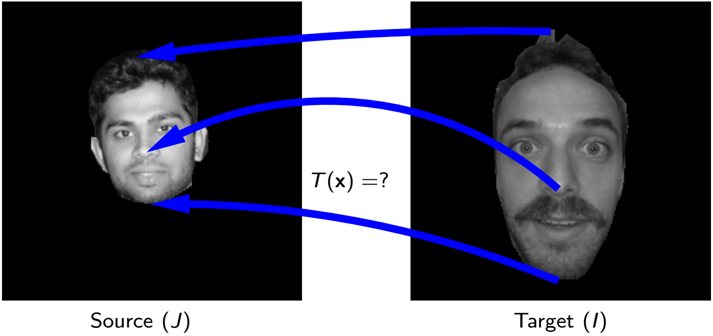
June 2, 2010

Outline

- Introduction to image registration
- Diffusion-based non-rigid image registration
- Traditional Finite Difference (FD) implementation
- Our proposed Finite Element Method (FEM) implementation
- Results
- Conclusion
- Acknowledgments

Affine registration



$$T(\mathbf{x}) = \mathbf{x} + \mathbf{U}(\mathbf{x}) = \begin{bmatrix} 1 + a_1 & a_3 \\ a_2 & 1 + a_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a_5 \\ a_6 \end{bmatrix}$$

Solve: $\underset{a_1,\ldots,a_6}{\operatorname{argmin}} \sum_{\mathbf{x}} (I(\mathbf{x}) - J(\mathbf{x} + \mathbf{U}(\mathbf{x})))^2$ where $\mathbf{U} = [U_x \; U_y]$

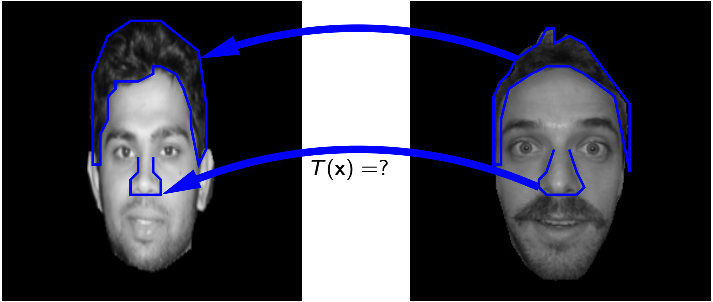
Affine registration

Registered

Target (I)

$$\mathbf{U}(\mathbf{x}) := \begin{bmatrix} 0.076 & 0.064 \\ -0.106 & 0.390 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -0.836 \\ 45.970 \end{bmatrix}$$

Non-rigid registration



Source (J) Target (I)

$$T(\mathbf{x}) = \mathbf{x} + \mathbf{U}(\mathbf{x}) \qquad \mathbf{U} : \Omega \rightarrow \mathbb{R}^2$$

$$E_{data}[\mathbf{U}] = \int_{\Omega} (I(\mathbf{x}) - J(\mathbf{x} + \mathbf{U}(\mathbf{x})))^2 \, d\mathbf{x}$$

$$\mathbf{U}^* = \text{argmin } E_{data}[\mathbf{U}]$$

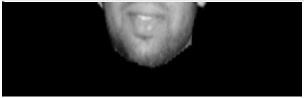
Non-rigid registration

Variational minimization of E_{data} by gradient descent

- Calculus of variations gives the **variational derivative** of E_{data} as:

$$\frac{\delta E_{data}}{\delta \mathbf{U}} = 2[J(\mathbf{x}+\mathbf{U}(\mathbf{x})) - I(\mathbf{x})][\nabla J(\mathbf{x})|_{\mathbf{x}+\mathbf{U}(\mathbf{x})}]$$

Define $\mathbf{u}(\mathbf{x}) = -2\,\epsilon\,[J(\mathbf{x}+\mathbf{U}(\mathbf{x})) - I(\mathbf{x})][\nabla J(\mathbf{x})|_{\mathbf{x}+\mathbf{U}(\mathbf{x})}]$ as the update field



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- Compositive update rule [Stefanescu et al., 2004]: Replace $\nabla J(\mathbf{x})|_{\mathbf{x}+\mathbf{U}(\mathbf{x})}$ (resampled gradient) with $\nabla J(\mathbf{x}+\mathbf{U}(\mathbf{x}))$(gradient of resampled image)

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Target (I)

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- Gradient descent scheme to minimize E_{data}:

$$\mathbf{u}^k(\mathbf{x}) = -2\,\epsilon\,[J(\mathbf{x}+\mathbf{U}^k(\mathbf{x})) - I(\mathbf{x})][\nabla J(\mathbf{x}+\mathbf{U}^k(\mathbf{x}))]$$
$$\mathbf{U}^{k+1}(\mathbf{x}) = \mathbf{U}^k(\mathbf{x}+\mathbf{u}^k(\mathbf{x})) + \mathbf{u}^k(\mathbf{x})$$

$$E_{data}[\mathbf{U}] = \int_{\mathbf{\Omega}} (I(\mathbf{x}) - J(\mathbf{x}+\mathbf{U}(\mathbf{x})))^2\, d\mathbf{x}$$
$$\mathbf{U}^* = \text{argmin}\ E_{data}[\mathbf{U}]$$

Non-rigid registration

Source (J)

FAIL

Target (I)

- Ill-posed minimization problem, the estimated displacement field $\mathbf{U}^*(\mathbf{x})$ moves each pixel independently !
- More constraints need to be added during the minimization of $E_{data}[\mathbf{U}]$

Non-rigid registration with smoothing

Source (J)

Registered

Target (I)

- Enforce the **smoothness** constraint, i.e., smooth the update field $\mathbf{u}^k(\mathbf{x})$ and displacement field $\mathbf{U}^k(\mathbf{x})$ at each step k [Stefanescu et al., 2004]. This can be achieved by **minimizing**:

$$E_{smooth}[v_l^k] = \int_\Omega (v_l^k - v_l^{*k})^2 + \alpha\, \Psi(\left|\left|\nabla v_l^k\right|\right|^2) d\mathbf{x}$$

$$\forall\, l \in \{x, y\} \quad \forall\, v \in \{U_x, U_y, u_x, u_y\}$$

Minimization of smoothing energy E_{smooth}

$$v_\alpha = \text{argmin } E_{smooth}[v] = \text{argmin } \int_\Omega (v - v^*)^2 + \alpha \Psi(||\nabla v||^2) d\mathbf{x}$$

where v_α is the smoothed displacement (update) field

Setting the variational derivative of E_{smooth} equal to zero we get the elliptic version of the diffusion Partial Differential Equation (PDE):

$$\frac{\delta E_{smooth}}{\delta v} = 2[v - v^* - \alpha\, \mathbf{div}(\Psi^{'}(||\nabla v||^2)\nabla v)] = 0$$

Alternatively, using variational calculus we can also set the **integral extremum** condition to zero:

$$L(v, h) = \int_\Omega \left[(v - v^*)h + \alpha \Psi^{'}(||\nabla u||^2)\nabla v . \nabla h \right] d\mathbf{x} = 0 \quad \forall h \in \mathcal{D}_1(\Omega)$$

Numerical methods for minimization of E_{smooth}

Finite differences to solve the **diffusion equation** [Stefanescu et al., 2004]:

- Consider an UNIFORM discretization of a $L \times W$ grid:

Finite Element Method to solve the **integral equation** [Popuri et al., 2010]:

- Consider a NON-UNIFORM discretization of the $L \times W$ grid:

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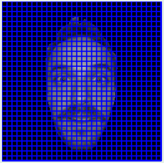
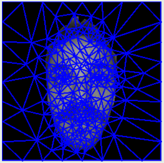
Uniform vs Non-Uniform grids

Template



Uniform grid



Non-Uniform grid

- Uniform discretization: fixed spacing of 1 pixel, total of $\mathbf{L} \times \mathbf{W}$ points
- Non-uniform discretization: less nodes in homogeneous regions and more nodes in regions with features, total nodes $\mathbf{M} << \mathbf{L} \times \mathbf{W}$

Numerical methods for minimization of E_{smooth}

Finite differences to solve the **diffusion equation** [Stefanescu et al., 2004]:

- Consider an UNIFORM discretization of a $L \times W$ grid:

$$v_{ij} - v^*_{ij} - \alpha\,[\mathbf{div}(\Psi'(||\nabla v||^2)\nabla v]_{ij} = 0$$

Discretize $[\mathbf{div}(.)]_{ij}$, use a semi-implicit **AOS** scheme [Weickert et al., 1998] and re-arrange:

$$\mathbf{v} = \frac{1}{2} \sum_{l \in \{x,y\}} (Id - 2\alpha(A_l))^{-1}\mathbf{v}^*$$

where $\mathbf{v} = \{v_{11}, v_{12}, \ldots, v_{LW}\}$, A_l is a matrix of constant coefficients

The penalizer

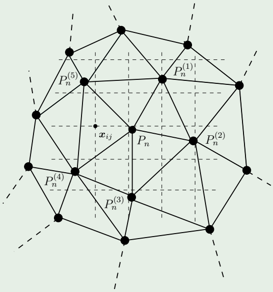
$$\Psi'(.) = \begin{cases} D(\mathbf{x}) & \text{displacement field} \\ 1 - k(\mathbf{x}) & \text{update field} \end{cases}$$

is constant

- $D(\mathbf{x})$ is the scalar inhomogeneous stiffness field
- $k(\mathbf{x}) = \exp\left(\frac{-c}{\left(\frac{\|\nabla J\|}{\lambda}\right)}\right)$ is the confidence field computed on the source image J
- Further, for the update field smoothing is performed on $\hat{v} = \frac{v}{1-k(\mathbf{x})}$ instead of v

Numerical methods for minimization of E_{smooth}

The nodal basis functions



$$\phi_n(\mathbf{x}) = \begin{cases} & \text{is linear within each triangle } \delta_{ij} \\ 1 & \text{at each node } P_n \\ 0 & \text{at every other node } P_m \neq P_n \end{cases}$$

The integral condition

$$\int_\Omega \left[(v - v^*)h + \alpha \Psi'(||\nabla u||^2)\nabla v . \nabla h\right] d\mathbf{x} = 0$$
$\forall h \in \mathcal{D}_1(\Omega)$

Approximate $v = \sum_{n=1}^{N} v(P_n)\phi_n$ and choosing $h = \phi_m$ we get

$$\sum_{n=1}^{N} (v(P_n) - v^*(P_n)) \int_\Omega \phi_n \phi_m \, d\mathbf{x} + \alpha \sum_{n=1}^{N} \int_\Omega \Psi'(.) \nabla\phi_n . \nabla\phi_m \, d\mathbf{x} = 0 \qquad m, n = \{1, 2, \ldots, N\}$$

Numerical methods for minimization of E_{smooth}

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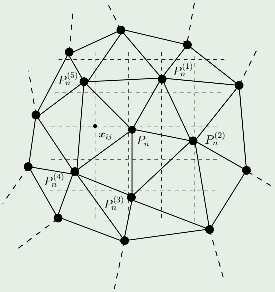

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Precomputation of the integrals

- The integrals $\int_\Omega \phi_n \phi_m$, $\int_\Omega \nabla\phi_n \nabla\phi_m$ are precomputed **analytically**

discretization of the $L \times W$ grid:

Approximate $v = \sum_{n=1}^{N} v(P_n)\phi_n$ and choosing $h = \phi_m$ we get

$$\sum_{n=1}^{N} (v(P_n) - v^*(P_n)) \int_\Omega \phi_n \phi_m \, d\mathbf{x} + \alpha \sum_{n=1}^{N} \int_\Omega \Psi'(.) \nabla\phi_n . \nabla\phi_m \, d\mathbf{x} = 0 \qquad m, n = \{1, 2, \ldots, N\}$$

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where $\mathbf{v} = \{v_{11}, v_{12}, \ldots, v_{LW}\}$, A_l is a matrix of constant coefficients

- We have to solve a system of $\mathbf{L} \times \mathbf{W}$ linear equations

Finite Element Method to solve the **integral equation** [Popuri et al., 2010]:

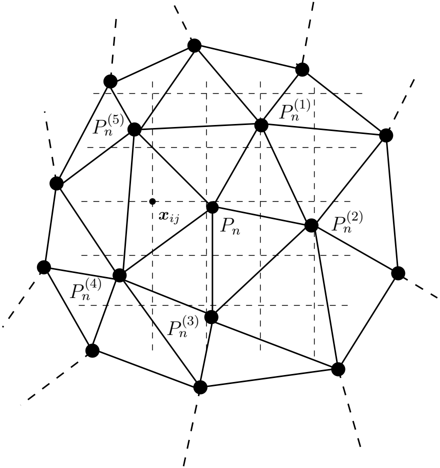
- Consider a NON-UNIFORM discretization of the $L \times W$ grid:

Approximate $v = \sum_{n=1}^{N} v(P_n)\phi_n$ and choosing $h = \phi_m$ we get

$$\sum_{n=1}^{N}(v(P_n) - v^*(P_n))\int_\Omega \phi_n\phi_m\,\mathbf{dx} + \alpha\sum_{n=1}^{N}\int_\Omega \Psi'(.)\nabla\phi_n.\nabla\phi_m\,\mathbf{dx} = 0 \qquad m, n = \{1, 2, \ldots, N\}$$

- We have to solve a system of $\mathbf{N} << \mathbf{L} \times \mathbf{W}$ linear equations. **Hence, our proposed method is much faster !**

“Implicit” update field smoothing



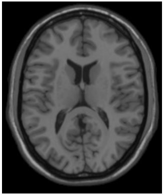
- Updates are computed at the nodes $\mathbf{u}(P_n)$ by taking a weighted average of the updates at the neighboring pixels $\mathbf{u}_{ij}$:

$$\mathbf{u}^k(P_n) = \frac{1}{\sum_{ij} \lambda_{ij}} \sum_{ij} \lambda_{ij} \mathbf{u}_{ij}^k$$

where λ_{ij} represents the barycentric coordinate of the pixel $\mathbf{x}_{ij}$ with respect to the node P_n

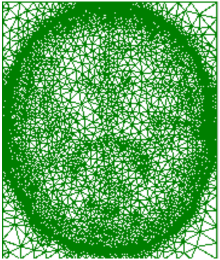
- Thus, we do NOT need to perform the more expensive diffusion based smoothing of the update field

Non-Uniform grid generation



Input image

Feature image



Final Mesh

Halftoned image

- Given an input 2D image $f(x, y)$ compute the feature map:

$$\sigma(x, y) = \left(\frac{G(x, y)}{K}\right)$$

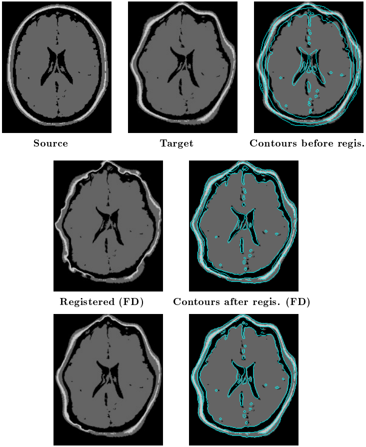
  where
  $G(x, y) = \max |f_{\theta}^{''}(x, y)| \quad \theta \in [0, 2\pi]$
  and K is a normalizing constant

- Halftone the feature image $\sigma(x, y)$ to obtain a binary image

- Input the locations of the white pixels in the binary image as initial grid nodes to a Delaunay grid generation algorithm

- Refine the grid generated from the above step to obtain the final image adapted non-uniform grid

Results



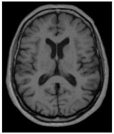
Source Target Contours before regis.

Registered (FD) Contours after regis. (FD)

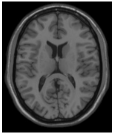
Registered (FEM) Contours after regis. (FEM)

- $D(\mathbf{x}) = 0.1$ in the ventricle region and $D(\mathbf{x}) = 0.5$ in the rest of the brain
- 13.07 sec (FEM), 57.06 sec (FD)

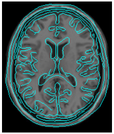
Results



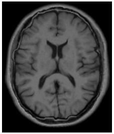
Source



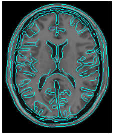
Target



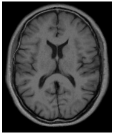
Contours before regis.



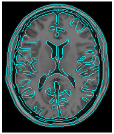
Registered (FD)



Contours after regis. (FD)



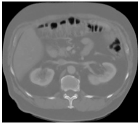
Registered (FEM)



Contours after regis. (FEM)

- $D(\mathbf{x}) = 0.01$ in the ventricle region and to $D(\mathbf{x}) = 0.1$ in the rest of the brain
- 21.31 sec (FEM), 112.66 sec (FD)

Results



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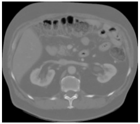
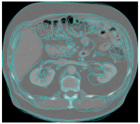
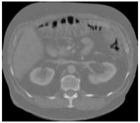


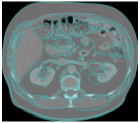
Target



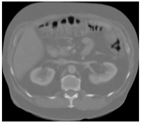
Contours before regis.



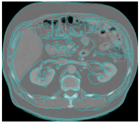
Registered (FD)



Contours after regis. (FD)



Registered (FEM)



Contours after regis. (FEM)

- 65.59 sec (FEM), 303.72 sec (FD)

Conclusions and Future work

- A fast Finite Element Method based non-rigid registration method that employed a grid with variable resolution was presented
- Only 2D images were considered in this paper, it can be easily extended to 3D images
- We intend to explore the possibility of learning the stiffness field $D(\mathbf{x})$ from a set of training images.

- Supervisors Dana Cobzas and Martin Jägersand
- Members of the Computer Vision group: Neil Birkbeck, David Lovi at the University of Alberta
- Dr. Vickie Baracos from the Department of Oncology, Cross Cancer Institute at the University of Alberta for providing CT data
- Dr. Albert Murtha from the Department of Oncology, Cross Cancer Institute at the University of Alberta for clinical advice

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