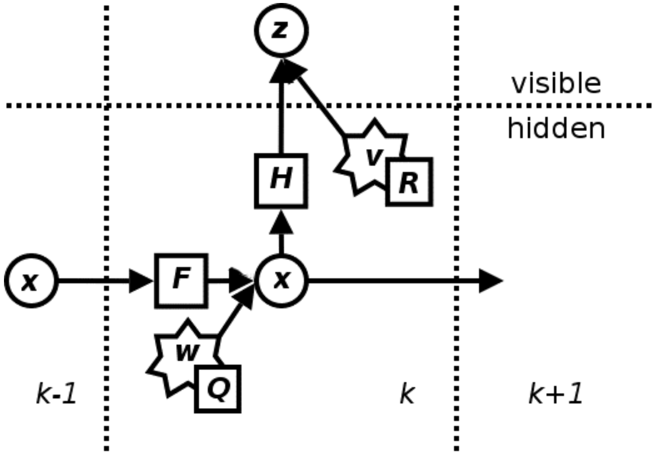
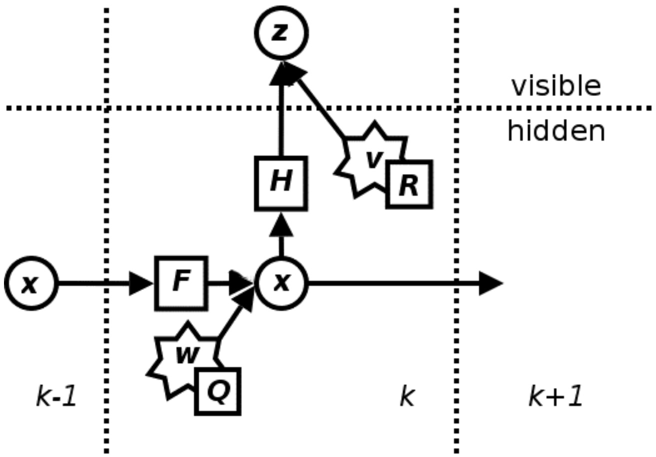
Kalman Filters



Derivation of Kalman Filter equations

Mark Fiala – CRV'09 Tutorial Day
May 24/2009

Kalman Filters



Predict

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{F}_k \hat{\mathbf{x}}_{k-1|k-1} + \mathbf{B}_k \mathbf{u}_k$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_k \mathbf{P}_{k-1|k-1} \mathbf{F}_k^{\mathsf{T}} + \mathbf{Q}_k$$

Update

$$\mathbf{S}_k = \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^{\mathsf{T}} + \mathbf{R}_k$$

$$\mathbf{K}_k = \mathbf{P}_{k|k-1} \mathbf{H}_k^{\mathsf{T}} \mathbf{S}_k^{-1}$$

$$\tilde{\mathbf{y}}_k = \mathbf{z}_k - \mathbf{H}_k \hat{\mathbf{x}}_{k|k-1}$$

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \tilde{y}_k$$

$$\mathbf{P}_{k|k} = (I - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_{k|k-1}$$

Normal Distribution - Gaussian

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Normal distribution

From Wikipedia, the free encyclopedia

In probability theory and statistics, the **normal distribution** or **Gaussian distribution** is a continuous probability distribution that describes data that clusters around a mean or average. The graph of the associated probability density function is bell-shaped, with a peak at the mean, and is known as the **Gaussian function** or **bell curve**.

The normal distribution can be used to describe, at least approximately, any variable that tends to cluster around the mean. For example, the heights of adult males in the United States are roughly normally distributed, with a mean of about 70 inches. Most men have a height close to the mean, though a small number of outliers have a height significantly above or below the mean. A histogram of male heights will appear similar to a bell curve, with the correspondence becoming closer if more data is used.

For theoretical reasons (such as the central limit theorem), any variable that is the sum of a large number of independent factors is likely to be normally distributed. For this reason, the normal distribution is used throughout statistics, natural science, and social science[1] as a simple model for complex phenomena. For example, the observational error in an experiment is usually assumed to follow a normal distribution, and the propagation of uncertainty is computed using this assumption.

The probability density function for a normal distribution is given by the formula

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),$$

where μ is the mean, σ is the standard deviation (a measure of the "width" of the bell), and exp denotes the exponential function. For a mean of 0 and a standard deviation of 1, this formula simplifies to

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2},$$

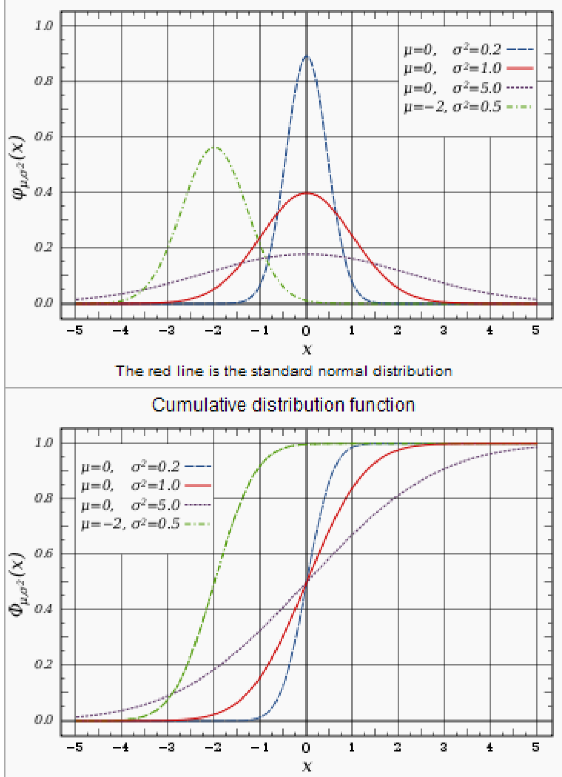
which is known as the **standard normal distribution**. When properly scaled and translated, the corresponding cumulative distribution function is known as the error function.

The Gaussian distribution is named for Carl Friedrich Gauss, who used it to analyze astronomical data[2], and defined the formula for its probability density function.

Contents [hide]

Normal

Probability density function



The red line is the standard normal distribution

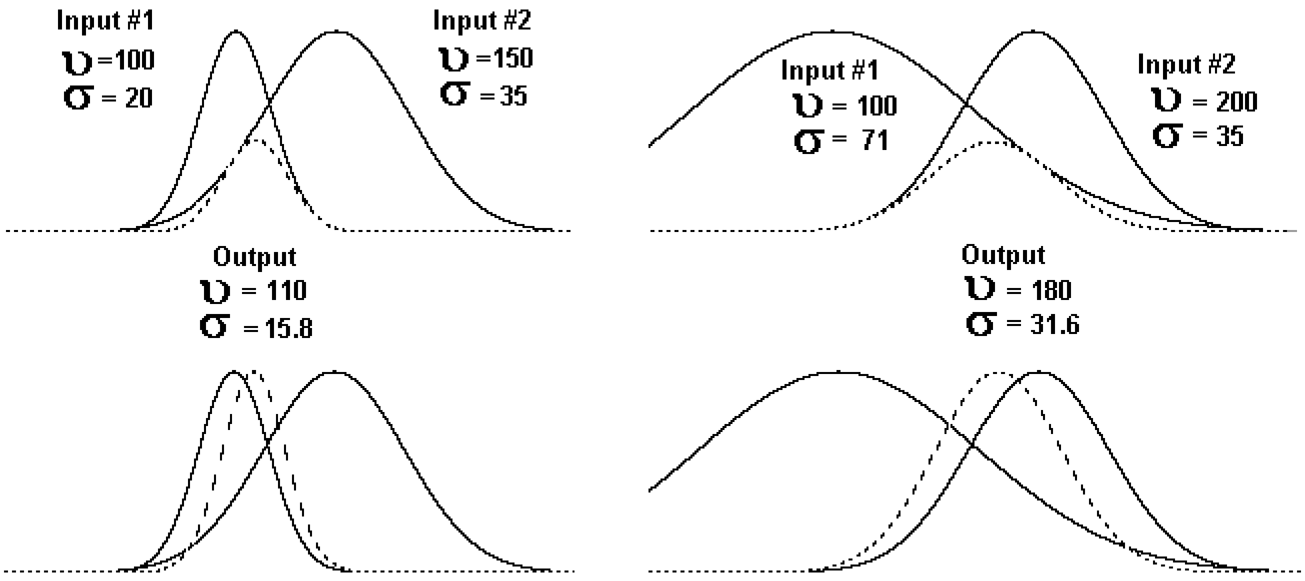
Cumulative distribution function

Colors match the image above

- **Gaussian models normal distribution**
- **1-D parameters: mean, sigma**
- **2-D parameters: mean vector, covariance matrix**

1-D Gaussians

- **The Kalman filter is based on manipulating gaussian approximations of probability density functions (PDF’s).**
- **Useful property of gaussian functions is that multiplying two gaussian functions yields a third gaussian.**



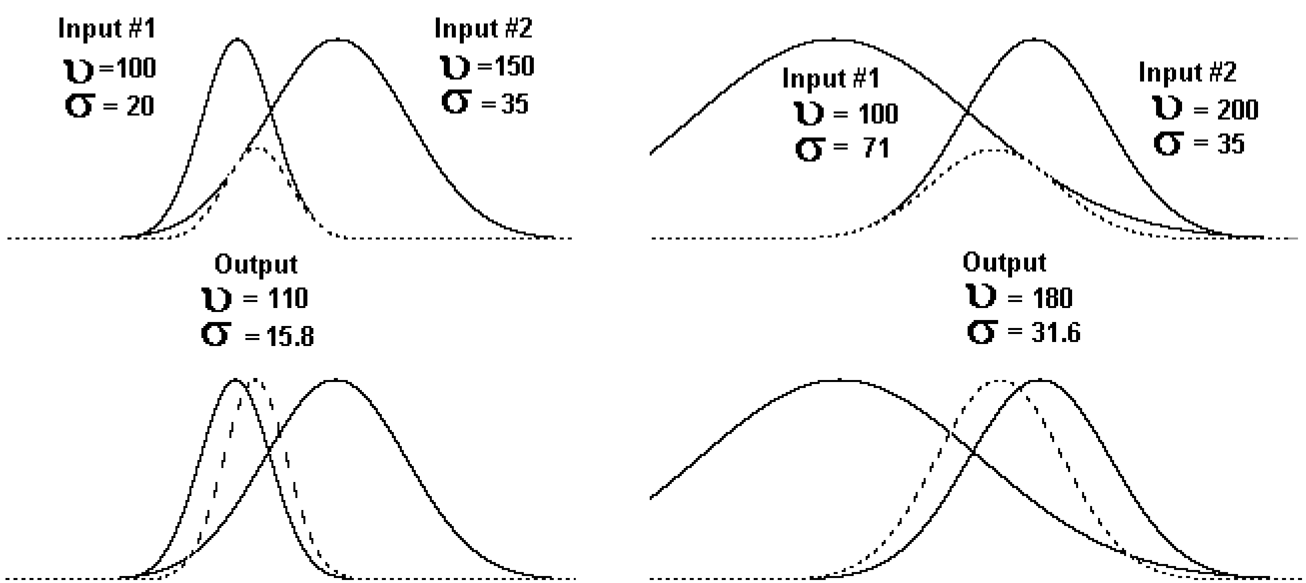
Hardware designers select one or the other based on situation, usually favor separate enables for timing reasons

Multiplying two 1-D Gaussians

$$PDF_1 = exp{-}(\frac{x-a}{2e^2})^2, PDF_2 = exp{-}(\frac{x-b}{2f^2})^2$$

$$PDF_3 = PDF_1 \cdot PDF_2 = exp{-}(\frac{x-\mu}{2\sigma^2})^2$$

$$\mu = a\frac{f^2}{e^2+f^2} + b\frac{e^2}{e^2+f^2}, \qquad \sigma^2 = \frac{1}{\frac{1}{e^2}+\frac{1}{f^2}}$$



Deriving multiplying two 1-D Gaussians

$$exp-(\frac{x-a}{2e^2})^2 exp-(\frac{x-b}{2f^2})^2 = exp-\frac{1}{2}[\frac{x-a}{e^2})^2 + \frac{x-b}{f^2})^2]$$

$$= exp-\frac{1}{2}[\frac{x^2-2ax-a^2}{e^2} + \frac{x^2-2bx+b^2}{f^2}]$$

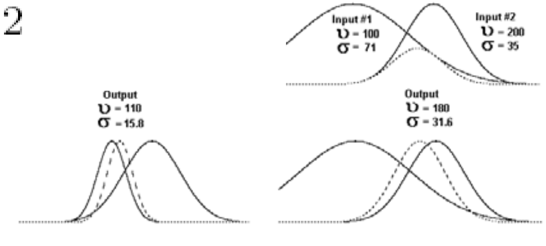
$$= exp-\frac{1}{2}[(\frac{1}{e^2}+\frac{1}{f^2})x^2 - 2(\frac{a}{e^2}+\frac{b}{f^2})x + (\frac{a^2}{e^2}+\frac{b^2}{f^2})]$$

$$= exp-\frac{1}{2}[\frac{x^2-2\mu x-\mu^2}{\sigma^2}]$$

Matching terms, $\frac{1}{\sigma^2} = (\frac{1}{e^2}+\frac{1}{f^2})$ and $\frac{\mu}{\sigma^2} = (\frac{a}{e^2}+\frac{b}{f^2})$

$$\sigma^2 = \frac{1}{\frac{1}{e^2}+\frac{1}{f^2}}$$

$$\mu = \frac{1}{\frac{1}{e^2}+\frac{1}{f^2}}(\frac{a}{e^2}+\frac{b}{f^2}) = a\frac{f^2}{e^2+f^2} + b\frac{e^2}{e^2+f^2}$$



Deriving multiplying three 1-D Gaussians

$$exp-(\frac{x-a}{2e^2})^2 exp-(\frac{x-b}{2f^2})^2 exp-(\frac{x-c}{2g^2})^2$$

$$= exp-\frac{1}{2}[\frac{x-a}{e^2})^2 + \frac{x-b}{f^2})^2 + \frac{x-c}{g^2})^2]$$

$$= exp-\frac{1}{2}[\frac{x^2-2ax-a^2}{e^2} + \frac{x^2-2bx+b^2}{f^2} + \frac{x^2-2cx+c^2}{g^2}]$$

$$= exp-\frac{1}{2}[(\frac{1}{e^2}+\frac{1}{f^2}+\frac{1}{g^2})x^2 - 2(\frac{a}{e^2}+\frac{b}{f^2}+\frac{c}{g^2})x + (\frac{a^2}{e^2}+\frac{b^2}{f^2}+\frac{c^2}{g^2})]$$

$$= exp-\frac{1}{2}[\frac{x^2-2\mu x-\mu^2}{\sigma^2}]$$

Matching terms, $\frac{1}{\sigma^2} = (\frac{1}{e^2}+\frac{1}{f^2})$ and $\frac{\mu}{\sigma^2} = (\frac{a}{e^2}+\frac{b}{f^2})$

$$\sigma^2 = \frac{1}{\frac{1}{e^2}+\frac{1}{f^2}}$$

$$\mu = \frac{1}{\frac{1}{e^2}+\frac{1}{f^2}}(\frac{a}{e^2}+\frac{b}{f^2}) = a\frac{f^2}{e^2+f^2} + b\frac{e^2}{e^2+f^2}$$

Multiplying 1-D Gaussians

$$PDF_1 = exp{-}(\frac{x-a}{2e^2})^2 \quad PDF_2 = exp{-}(\frac{x-b}{2f^2})^2$$

$$PDF_3 = exp{-}(\frac{x-c}{2g^2})^2$$

Multiplying 2 gaussians

$$PDF_1 \cdot PDF_2 = exp{-}(\frac{x-\mu}{2\sigma^2})^2$$

$$\mu = a\frac{f^2}{e^2+f^2} + b\frac{e^2}{e^2+f^2}, \qquad \sigma^2 = \frac{1}{\frac{1}{e^2}+\frac{1}{f^2}}$$

Multiplying 3 gaussians

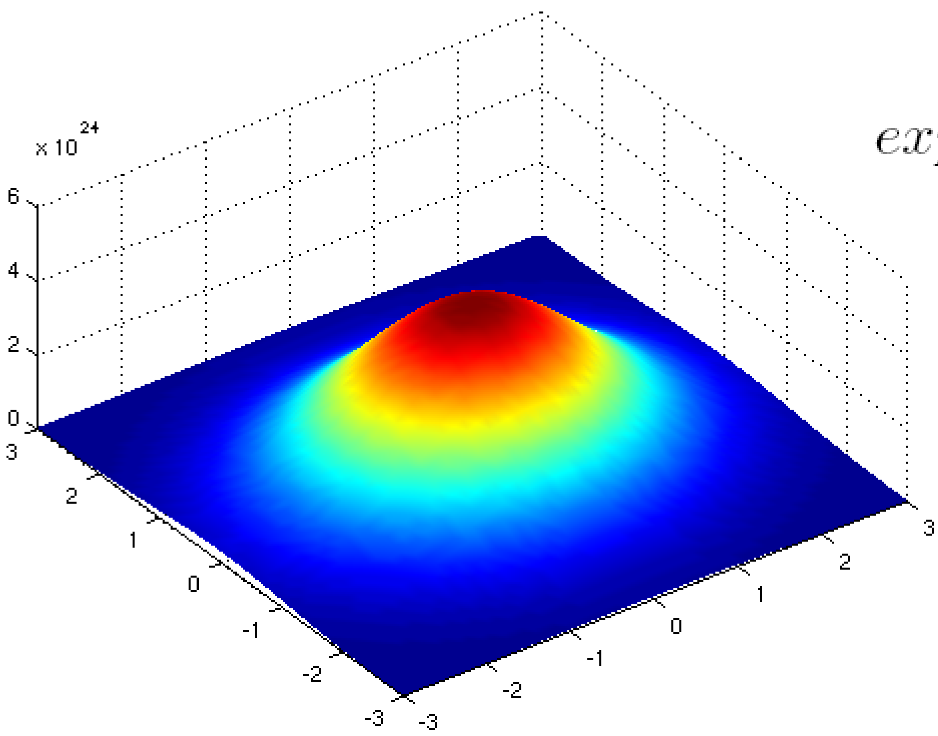
$$\mu = a\frac{f^2+g^2}{e^2+f^2+g^2} + b\frac{e^2+g^2}{e^2+f^2+g^2} + c\frac{e^2+f^2}{e^2+f^2+g^2},$$

$$\sigma^2 = \frac{1}{\frac{1}{e^2}+\frac{1}{f^2}+\frac{1}{g^2}}$$

Show 1d_gaussian_gui.exe

N-D Gaussians

- **Multi-dimensional gaussian can be created by putting gaussians on different orthogonal axes = multiplying with different variables**

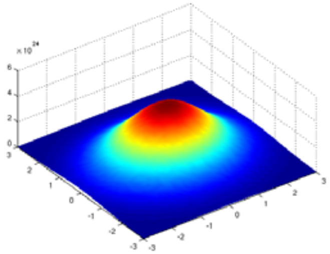


- **2-D example: One gaussian in X-axis, one in Y-axis**

$$exp-(\frac{(x-\mu_x)^2}{2\sigma_x^2} \cdot exp-(\frac{(y-\mu_y)^2}{2\sigma_y^2}$$

(Image from Wikipedia)

- **2-D gaussian aligned along X, Y axes**



$$exp-(\frac{(x-\mu_x)^2}{2\sigma_x^2} \cdot exp-(\frac{(y-\mu_y)^2}{2\sigma_y^2}$$

$$= exp-\frac{1}{2}\left(\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right)$$

$$= exp-\frac{1}{2}\left(\begin{bmatrix} x-\mu_x & y-\mu_y \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_x^2} & 0 \\ 0 & \frac{1}{\sigma_y^2} \end{bmatrix} \begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}\right)$$

$$= exp-\frac{1}{2}\left([\mathbf{X}-\mathbf{U}]^t \Sigma^{-1} [\mathbf{X}-\mathbf{U}]\right)$$

$$Where \quad \mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}, \mathbf{U} = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix}, \Sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma_x^2} & 0 \\ 0 & \frac{1}{\sigma_y^2} \end{bmatrix}$$

- **Use rotation matrix R to align along arbitrary axes**

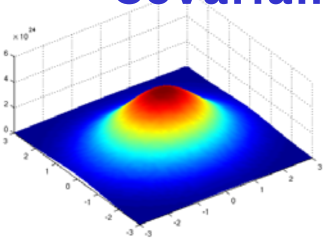
$$PDF = exp-\frac{1}{2}\left([\mathbf{X}-\mathbf{U}]^t \mathbf{R}^t \Sigma^{-1} \mathbf{R} [\mathbf{X}-\mathbf{U}]\right)$$

$$= exp-\frac{1}{2}\left([\mathbf{X}-\mathbf{U}]^t \mathbf{C} [\mathbf{X}-\mathbf{U}]\right)$$

- **Matrix C is not diagonal as is Σ**

Justification of N-D gaussians

- **Covariance matrix of an 2-D data set**



$$Cov = \begin{bmatrix} \sum (x_i - \mu_x)(x_i - \mu_x) & \sum (x_i - \mu_x)(y_i - \mu_y) \\ \sum (x_i - \mu_x)(y_i - \mu_y) & \sum (y_i - \mu_y)(y_i - \mu_y) \end{bmatrix}$$

- **SVD**

$A=UDV^t$ D = diagonal
SVD of $A=S^tS$: U=V
$A=UDU^t$

- **Covariance matrix of an N-D data set**

$$Cov = \mathbf{S}^t\mathbf{S} = \mathbf{R}^t\Sigma\mathbf{R}$$

$$Cov^{-1} = \mathbf{R}^t\Sigma^{-1}\mathbf{R} = \mathbf{C}$$

$$\Sigma = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix}, \Sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma_x^2} & 0 \\ 0 & \frac{1}{\sigma_y^2} \end{bmatrix}$$

- **We need $1/\sigma^2$ form**

Therefore

-a covariance matrix can be rotated to axes where Σ is a diagonal matrix

-a covariance matrix can be represented as an N-D shape of orthogonal gaussians

$$PDF = exp - \frac{1}{2}\left([\mathbf{X}-\mathbf{U}]^t\mathbf{R}^t\Sigma^{-1}\mathbf{R}[\mathbf{X}-\mathbf{U}]\right)$$

$$= exp - \frac{1}{2}\left([\mathbf{X}-\mathbf{U}]^t\mathbf{C}[\mathbf{X}-\mathbf{U}]\right)$$

- **PDF=k =ellipse in 2-D, =ellipsoid in 3-D**

Linear Functions and PDF's

What happens to a PDF when passing through a linear function (matrix operation)?

- **Input variable X – has mean U_x and covariance Cov_x**
- **Linear function Y = A X**
- **Output variable Y – has mean U_y and covariance Cov_y**

$$U_y = A\,U_x$$

$$Cov_y = A\,Cov_x\,A^t$$

$$PDF = exp - \frac{1}{2}\left([\mathbf{X}-\mathbf{U}]^t \mathbf{R^t} \Sigma^{-1} \mathbf{R} [\mathbf{X}-\mathbf{U}]\right)$$
$$= exp - \frac{1}{2}\left([\mathbf{X}-\mathbf{U}]^t \mathbf{C} [\mathbf{X}-\mathbf{U}]\right)$$

$$PDF = exp - \frac{1}{2}\left([\mathbf{X^t}-\mathbf{U^t}]\mathbf{C}[\mathbf{X}-\mathbf{U}]\right)$$

- **Expanded form – useful for multiplication**

$$PDF = exp - \frac{1}{2}\left([\mathbf{X^t}-\mathbf{U^t}]\mathbf{C}[\mathbf{X}-\mathbf{U}]\right)$$
$$= exp - \frac{1}{2}\left([\mathbf{X^tC}-\mathbf{U^tC}][\mathbf{X}-\mathbf{U}]\right)$$
$$= exp - \frac{1}{2}\left(\mathbf{X^tCX}-\mathbf{U^tCX}-\mathbf{X^tCU}+\mathbf{U^tCU}\right)$$
$$= exp - \frac{1}{2}\left(\mathbf{X^tCX}-2\mathbf{X^tCU}+\mathbf{U^tCU}\right)$$

Multiplying N-D Gaussians

Each gaussian PDF has a mean (centroid) vector U and a covariance C^{-1}

Inputs $PDF_1 = exp - \frac{1}{2}\left([\mathbf{X}-\mathbf{U_1}]^t \mathbf{C_1} [\mathbf{X}-\mathbf{U_1}]\right)$ $PDF_2 = exp - \frac{1}{2}\left([\mathbf{X}-\mathbf{U_2}]^t \mathbf{C_2} [\mathbf{X}-\mathbf{U_2}]\right)$

Output $PDF_R = exp - \frac{1}{2}\left([\mathbf{X}-\mathbf{U_r}]^t \mathbf{C_r} [\mathbf{X}-\mathbf{U_r}]\right) = exp - \frac{1}{2}\left(\mathbf{X^t C_r X} - 2\mathbf{X^t C_r U_r} + \mathbf{U_r^t C_r U_r}\right)$

Multiplying 2 gaussians

$$PDF_1 \cdot PDF_2 = exp - \frac{1}{2}\left(\mathbf{X^t C_1 X} - 2\mathbf{X^t C_1 U_1} + \mathbf{U_1^t C_1 U_1}\right) \cdot exp - \frac{1}{2}\left(\mathbf{X^t C_2 X} - 2\mathbf{X^t C_2 U_2} + \mathbf{U_2^t C_2 U_2}\right)$$

$$= exp - \frac{1}{2}\left(\mathbf{X^t C_1 X} - 2\mathbf{X^t C_1 U_1} + \mathbf{U_1^t C_1 U_1} + \mathbf{X^t C_2 X} - 2\mathbf{X^t C_2 U_2} + \mathbf{U_2^t C_2 U_2}\right)$$

$$= exp - \frac{1}{2}\left(\mathbf{X^t [C_1 + C_2] X} - 2\mathbf{X^t [C_1 U_1 + C_2 U_2]} + \mathbf{U_1^t C_1 U_1} + \mathbf{U_2^t C_2 U_2}\right)$$

$$\mathbf{C_r} = \mathbf{C_1} + \mathbf{C_2}$$

$$C_r U_r = C_1 U_1 + C_2 U_2,$$

$$U_r = C_r^{-1}[C_1 U_1 + C_2 U_2]$$

Multiplying N-D Gaussians

Each gaussian PDF has a mean (centroid) vector U and a covariance C^{-1}

Inputs $PDF_1 = exp - \frac{1}{2}\left([\mathbf{X} - \mathbf{U_1}]^t \mathbf{C_1} [\mathbf{X} - \mathbf{U_1}]\right)$ $PDF_2 = exp - \frac{1}{2}\left([\mathbf{X} - \mathbf{U_2}]^t \mathbf{C_2} [\mathbf{X} - \mathbf{U_2}]\right)$

Output $PDF_R = exp - \frac{1}{2}\left([\mathbf{X} - \mathbf{U_r}]^t \mathbf{C_r} [\mathbf{X} - \mathbf{U_r}]\right) = exp - \frac{1}{2}\left(\mathbf{X^t C_r X} - 2\mathbf{X^t C_r U_r} + \mathbf{U_r^t C_r U_r}\right)$

$$\mathbf{C_r} = \mathbf{C_1} + \mathbf{C_2} \qquad U_r = C_r^{-1}[C_1 U_1 + C_2 U_2]$$

Rename covariance $P = C^{-1}$

Inputs $PDF_1 = exp - \frac{1}{2}\left([\mathbf{X} - \mathbf{U_1}]^t \mathbf{P_1^{-1}} [\mathbf{X} - \mathbf{U_1}]\right)$ $PDF_2 = exp - \frac{1}{2}\left([\mathbf{X} - \mathbf{U_2}]^t \mathbf{P_2^{-1}} [\mathbf{X} - \mathbf{U_2}]\right)$

Output $PDF_R = PDF_1 \cdot PDF_2 = exp - \frac{1}{2}\left([\mathbf{X} - \mathbf{U_r}]^t \mathbf{P_r^{-1}} [\mathbf{X} - \mathbf{U_r}]\right)$

$$P_r = (P_1^{-1} + P_2^{-1})^{-1} \qquad U_r = P_r(P_1^{-1} U_1 + P_2^{-1} U_2)^{-1}$$

Multiplying Gaussians

Multiplying 1-D gaussians

$$PDF_1 = exp - (\frac{x-a}{2e^2})^2 \quad PDF_2 = exp - (\frac{x-b}{2f^2})^2$$

$$PDF_3 = exp - (\frac{x-c}{2g^2})^2$$

$$PDF_1 \cdot PDF_2 = exp - (\frac{x-\mu}{2\sigma^2})^2$$

$$\mu = a\frac{f^2}{e^2+f^2} + b\frac{e^2}{e^2+f^2}, \qquad \sigma^2 = \frac{1}{\frac{1}{e^2}+\frac{1}{f^2}}$$

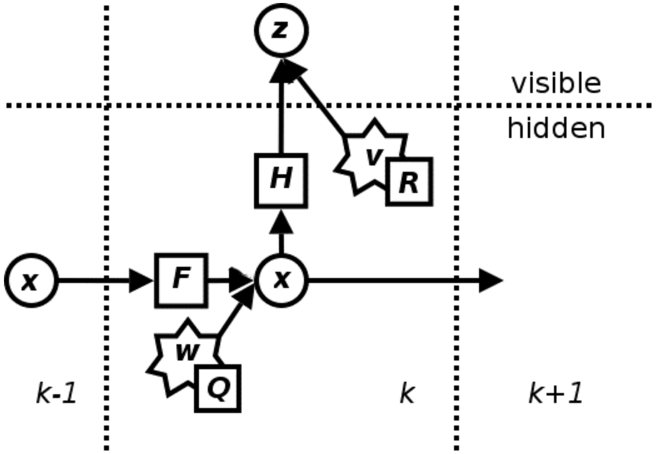
Multiplying N-D gaussians

Inputs $PDF_1 = exp - \frac{1}{2}([\mathbf{X}-\mathbf{U_1}]^t \mathbf{P_1}^{-1}[\mathbf{X}-\mathbf{U_1}]) \quad PDF_2 = exp - \frac{1}{2}([\mathbf{X}-\mathbf{U_2}]^t \mathbf{P_2}^{-1}[\mathbf{X}-\mathbf{U_2}])$

Output $PDF_R = PDF_1 \cdot PDF_2 = exp - \frac{1}{2}([\mathbf{X}-\mathbf{U_r}]^t \mathbf{P_r}^{-1}[\mathbf{X}-\mathbf{U_r}])$

$P_r = (P_1^{-1} + P_2^{-1})^{-1}$ $\qquad$ $U_r = P_r(P_1^{-1} U_1 + P_2^{-1} U_2)^{-1}$

Kalman Filters



Predict

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{F}_k \hat{\mathbf{x}}_{k-1|k-1} + \mathbf{B}_k \mathbf{u}_k$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_k \mathbf{P}_{k-1|k-1} \mathbf{F}_k^{\mathsf{T}} + \mathbf{Q}_k$$

Update

$$\mathbf{S}_k = \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^{\mathsf{T}} + \mathbf{R}_k$$

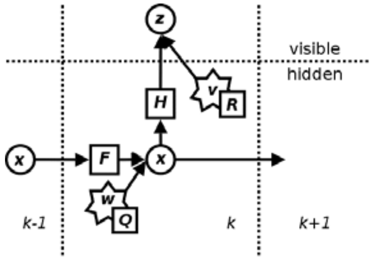
$$\mathbf{K}_k = \mathbf{P}_{k|k-1} \mathbf{H}_k^{\mathsf{T}} \mathbf{S}_k^{-1}$$

$$\tilde{\mathbf{y}}_k = \mathbf{z}_k - \mathbf{H}_k \hat{\mathbf{x}}_{k|k-1}$$

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \tilde{y}_k$$

$$\mathbf{P}_{k|k} = (I - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_{k|k-1}$$

Kalman Filters



Predict

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{F}_k \hat{\mathbf{x}}_{k-1|k-1} + \mathbf{B}_k \mathbf{u}_k$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_k \mathbf{P}_{k-1|k-1} \mathbf{F}_k^{\mathsf{T}} + \mathbf{Q}_k$$

Linear functions and PDF’s

U_y = A U_x

Cov_y = ACov_x A^t

Update

$$\mathbf{S}_k = \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^{\mathsf{T}} + \mathbf{R}_k$$

$$\mathbf{K}_k = \mathbf{P}_{k|k-1} \mathbf{H}_k^{\mathsf{T}} \mathbf{S}_k^{-1}$$

$$\tilde{\mathbf{y}}_k = \mathbf{z}_k - \mathbf{H}_k \hat{\mathbf{x}}_{k|k-1}$$

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \tilde{y}_k$$

$$\mathbf{P}_{k|k} = (I - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_{k|k-1}$$

Linear functions and PDF’s

U_y = A U_x

Cov_y = ACov_x A^t

Baye's Rule and Kalman Filters

Baye's Rule

- **Prob(A|B) ~= Prob(A)Prob(B|A)**
- **Output variable Y – has mean U_y and covariance Cov_y**

Kalman Filter – Find Prob(X) given measurements Z

- **X=state variables, Z=measurements**
- **Want Prob(X|Z)**
- **Prob(X|Z) ~= Prob(X)Prob(Z|X)** *(update eqns)*
- **X has normal distribution PDF given by mean = $\hat{\mathbf{x}}$ and Covar = P**
- **What is probability of observed Z given X?**
- **P(Z|X) has mean = Z and Covar = S**

$$\mathbf{S}_k = \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^{\mathsf{T}} + \mathbf{R}_k$$

Multiply two PDF's = Kalman Filter

Kalman Filter = Multiply two N-D Gaussians

Multiplying N-D gaussians

Inputs $PDF_1 = exp - \frac{1}{2}\left([\mathbf{X} - \mathbf{U_1}]^t \mathbf{P_1}^{-1} [\mathbf{X} - \mathbf{U_1}]\right)$ $PDF_2 = exp - \frac{1}{2}\left([\mathbf{X} - \mathbf{U_2}]^t \mathbf{P_2}^{-1} [\mathbf{X} - \mathbf{U_2}]\right)$

Output $PDF_R = PDF_1 \cdot PDF_2 = exp - \frac{1}{2}\left([\mathbf{X} - \mathbf{U_r}]^t \mathbf{P_r}^{-1} [\mathbf{X} - \mathbf{U_r}]\right)$

$P_r = (P_1^{-1} + P_2^{-1})^{-1}$ $U_r = P_r(P_1^{-1} U_1 + P_2^{-1} U_2)^{-1}$

Multiply two PDF's = Kalman Filter

- **Prob(X|Z) ~= Prob(X)Prob(Z|X)** *(update eqns)*
- **PDF_1=X has normal distribution PDF given by mean =** $\hat{\mathbf{X}}$ **and Covar = P**
- **PDF_2=Z has normal distribution PDF given by mean = Z and Covar = $HPH^t + R$**
- **$PDF_r = PDF_1 PDF_2$ = next iteration X,P**
- **$P_r = P_{next} = [P^{-1} + (HPH^t + R)]^{-1}$**
- **$U_r = X_{next} = [P^{-1} + (HPH^t + R)]^{-1} [P^{-1}X\hat{} + (HPH^t + R)^{-1} Z]$**

Kalman Filter = Multiply two N-D Gaussians

Multiply two PDF's = Kalman Filter

- $\mathbf{P_r = P_{next} = [P^{-1} + (HPH^t + R)]^{-1}}$
- $\mathbf{U_r = X_{next} = [P^{-1} + (HPH^t + R)]^{-1} \, [P^{-1}X\hat{} + (HPH^t + R)^{-1} Z}$ $\hat{\mathbf{x}}$

After some algebra and use of *inversion lemma*

$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}_k^{\mathsf{T}}\mathbf{S}_k^{-1}$$

$$\tilde{\mathbf{y}}_k = \mathbf{z}_k - \mathbf{H}_k\hat{\mathbf{x}}_{k|k-1}$$

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k\tilde{y}_k$$

$$\mathbf{P}_{k|k} = (I - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_{k|k-1}$$

EKF: Extended Kalman filter

- Allow non-linear functions (F, H)
- Apply functions to state

$$\mathbf{x}_k = f(\mathbf{x}_{k-1}, \mathbf{u}_k, \mathbf{w}_k)$$
$$\mathbf{z}_k = h(\mathbf{x}_k, \mathbf{v}_k)$$

- Apply jacobian to covariances
- Linearizing functions around current estimate