The 2D/3D Differential Optical Flow

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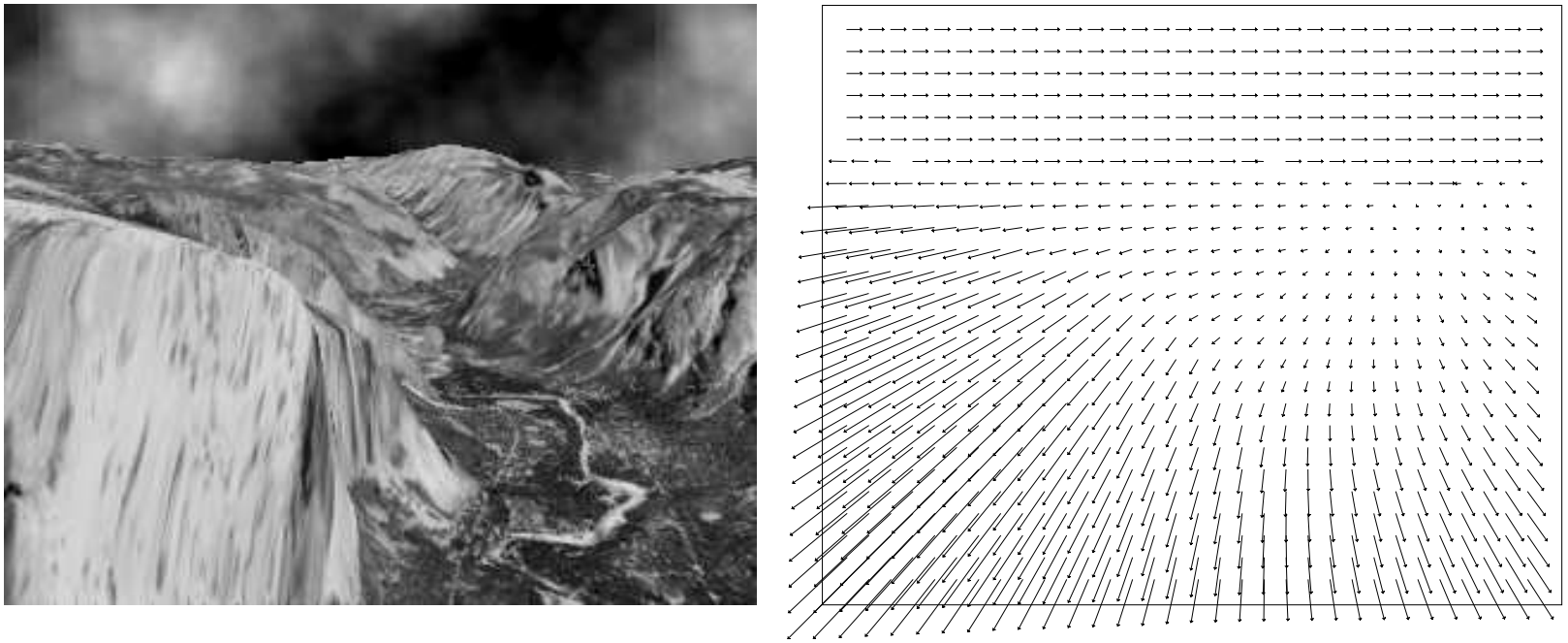
2D Optical Flow: An Example



(a) (b)

Figure 1: (a) The middle frame from the **Yosemite Fly-Through** sequence and (b) its correct flow field.

2D Optical Flow: An Overview

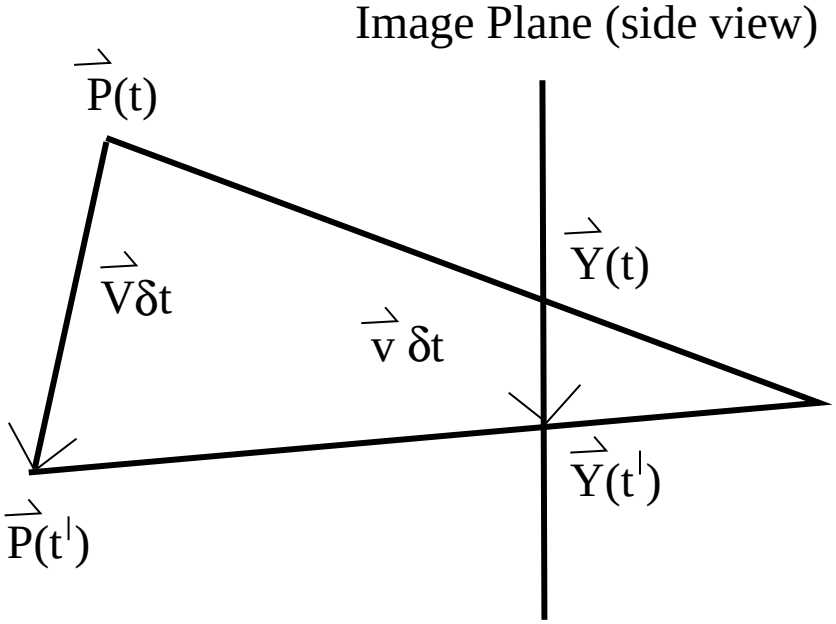


Figure 2: $\vec{V}$ is the 3D velocity of 3D point $\vec{P}(t) = (X, Y, Z)$. $\vec{v} = (u, v)$ is the 2D image of $\vec{V}$, i.e. $\vec{v}$ is the perspective projection of $\vec{V}$. If $\vec{P}(t)$ moves with displacement $\vec{V}\delta t$ to $\vec{P}(t')$ from time t to time t', then its image $\vec{Y}(t) = (x, y, f)$ moves with displacement $\vec{v}\delta t$ to $\vec{Y}(t') = (x', y', f)$ from times t to t'. f is the sensor focal. $\vec{v}$ is known as **image velocity** or **optical flow**.

The Image Velocity Equations

- The 3D instantaneous velocity $\vec{V} = (U, V, W)$ of a 3D point $\vec{P} = (X, Y, Z)$, where the sensor moves relative to $\vec{P}$ is

$$\vec{V} = (U, V, W) = -\vec{T} - \vec{\omega} \times \vec{P}, \tag{1}$$

  where $\vec{T} = (T_0, T_1, T_2)$ is the instantaneous sensor translation and $\vec{\omega} = (\omega_0, \omega_1, \omega_2)$ is the sensor's instantaneous rotation. The components of $\vec{V}$ can be written out in full as:

$$\begin{aligned} U &= -T_0 - \omega_1 Z + \omega_2 Y & (2)\\ V &= -T_1 - \omega_2 X + \omega_0 Z & (3)\\ W &= -T_2 - \omega_0 Y + \omega_1 X & (4) \end{aligned}$$

  For a rigid object under perspective projection we can write

$$x = f\frac{X}{Z}, \quad y = f\frac{Y}{Z} \quad \text{and} \quad z = f\frac{Z}{Z} = f, \tag{5}$$

where f is the focal length of the sensor. The time derivatives of (x, y, z), $(\dot{x}, \dot{y}, \dot{z}) = (u, v, 0)$, yield the two non-zero components of instantaneous image velocity, which can be written as

$$\dot{x} = u \quad = \quad f\left(\frac{\dot{X}}{Z} - \frac{X\dot{Z}}{Z^2}\right) \tag{6}$$

$$\dot{y} = v \quad = \quad f\left(\frac{\dot{Y}}{Z} - \frac{Y\dot{Z}}{Z^2}\right) \tag{7}$$

Substituting equations (2), (3) and (4) into equations (6) and (7) and using the definition of perspective projection in equation (5) we obtain the standard image velocity equations (see Longuet-Higgins and Prazdny, Proc. R. Soc. London B208, 1981):

$$u = \frac{1}{Z}\left(-fT_0 + xT_2\right) + \omega_0\left(\frac{xy}{f}\right) - \omega_1\left(f + \frac{x^2}{f}\right) + \omega_2 y$$

and

$$v = \frac{1}{Z}\left(-fT_1 + yT_2\right) + \omega_0\left(f + \frac{y^2}{f}\right) - \omega_1\left(\frac{xy}{f}\right) - \omega_2 x$$

- Given the sensor's instantaneous 3D translation and 3D rotation plus an image point's 2D coordinates (x, y) and its 3D depth, Z [hence we know X and Y as well via equations (5)] we can specify the correct 2D image velocity, (u, v), for this image point.
- Conversely, if we can measure (u, v) values at image points, we can recover the 3D sensor translation scaled by 3D depth and 3D sensor rotation. We can also recover "relative" depth at image points in a scene. So, while we can't know from a single monocular sensor that object 1 is 4m away while object 2 is 8m away (absolute depth) we can tell that object 1 is twice as close as object 2 (relative depth). Of course, with a binocular sensor setup we can recover absolute motion and depth parameters.

The 2D Aperture Problem

- We can estimate optical flow locally from spatial and temporal image intensity derivatives measured in local neighbourhoods.
- The aperture problem (Marr and Ullman 1981) is relevant when image velocities are measured locally.

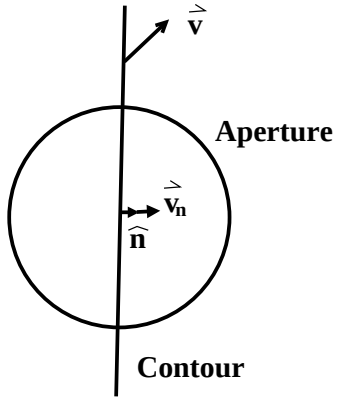


Figure 3: $\hat{n}$ is the unit normal vector in the normal velocity direction, i.e. it is perpendicular to the local contour structure and having length 1. Thus normal velocity $\vec{v}_n$ is the component of full velocity v projected in the normal direction $\hat{n}$, $(\vec{v} \cdot \hat{n})\hat{n} = \vec{v}_n$.

The Significance of the Aperture Problem

- So, due to the aperture problem, we can measure just $\vec{v}_n$ and not $\vec{v}_t$ (the tangential velocity) or $\vec{v}$ (the full velocity) locally.
- The aperture problem does not depend on the size of the aperture but rather on local contour structure: Is there enough local structure to recover full image velocity?

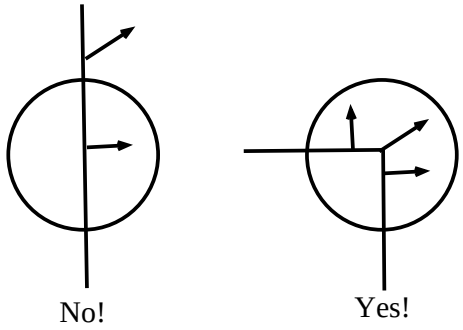
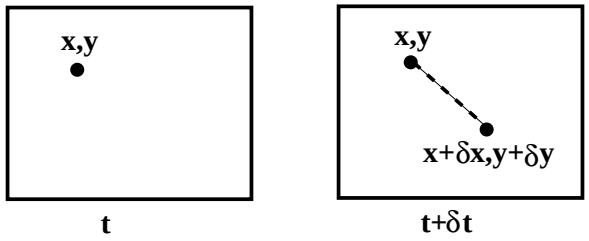


Figure 4: We can recover full image velocity in neighbourhoods of corner points but not in neighbourhoods when the local contour is straight.

The Motion Constraint Equation

- Assume $I(x, y, t)$ moves by δx, δy in time δt to $I(x + \delta x, y + \delta y, t + \delta t)$.



- Since $I(x, y, t)$ and $I(x + \delta x, y + \delta y, t + \delta t)$ are the images of the same point:

$$I(x, y, t) = I(x + \delta x, y + \delta y, t + \delta t).$$

- We can perform a 1^{st} order Taylor series expansion about $I(x, y, t)$:

$$I(x+\delta x, y+\delta y, t+\delta t) = I(x, y, t)+\frac{\partial I}{\partial x}\delta x+\frac{\partial I}{\partial y}\delta y+\frac{\partial I}{\partial t}\delta t+H.O.T.$$

The Motion Constraint Equation Continued:

- Because $I(x, y, t) = I(x + \delta x, y + \delta y, t + \delta t)$ we obtain:

$$\begin{aligned}
\frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t &= 0, \\
\frac{\partial I}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial I}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial I}{\partial t}\underbrace{\frac{\delta t}{\delta t}}_{=1} &= 0, \\
I_x u + I_y v + I_t &= 0 \text{ and} \\
\nabla I \cdot \vec{v} + I_t &= 0.
\end{aligned}$$

- Here $u = \frac{\delta x}{\delta t}$ and $v = \frac{\delta y}{\delta t}$ are the x and y components of image velocity and $I_x = \frac{\partial I}{\partial x}$, $I_y = \frac{\partial I}{\partial y}$ and $I_t = \frac{\partial I}{\partial t}$ are image intensity derivatives at $I(x, y, t)$.

Motion Constraint Line

- $\nabla I \cdot \vec{v} + I_t = 0$ is 1 equation in 2 unknowns (a line), the correct velocity is some unknown point on this line. The velocity with the smallest magnitude if the normal velocity $\vec{v}_n$.

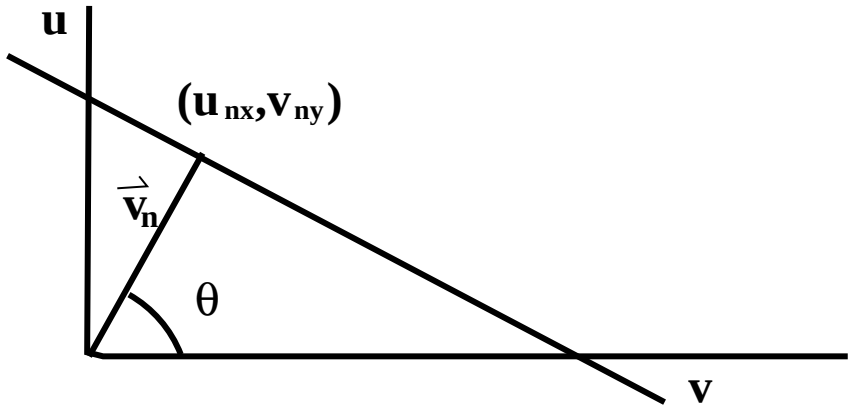
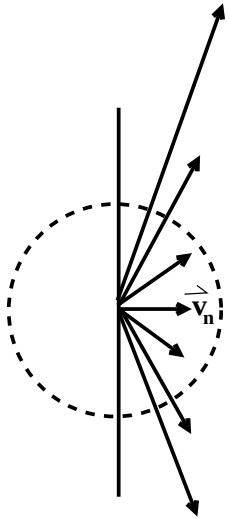


Figure 5: The motion constraint line. $\vec{v}_n = (u_{nx}, v_{ny})$ is the velocity with the smallest magnitude that is on this line.

The Relationship between Normal Velocity and the Motion Constraint Equation

- Consider a straight contour moving up/down and to the right with true full velocity $\vec{v}$. Since it is viewed through an aperture we can only see the motion perpendicular to the contour.



- Since the normal velocity is the smallest of all potential velocities it

is the point on the motion constraint line closest to the origin.

- We can compute v_n (the magnitude of $\vec{v}_n$) and $\hat{n}$ (its direction) and hence, the normal velocity, $\vec{v}_n = v_n \hat{n}$, very simply using the motion constraint equation $\nabla I \cdot \vec{v} = -I_t$ and the fact that $\vec{v} \cdot \hat{n} = v_n$ as:

$$\begin{aligned} \nabla I \cdot \vec{v} &= -I_t \\ \frac{\nabla I}{||\nabla I||_2} \cdot \vec{v} &= -\frac{I_t}{||\nabla I||_2} \\ \hat{n} \cdot \vec{v} &= v_n \end{aligned}$$

$$\Rightarrow \hat{n} = \frac{\nabla I}{||\nabla I||_2} \quad \text{and} \quad v_n = \frac{-I_t}{||\nabla I||_2}.$$

Resolving the Aperture Problem

- We need to impose additional constraints to recover the full velocity $\vec{v}$ everywhere.
- We could assume that each local image neighbourhood has constant velocity [Lucas and Kanade IJCAI1981]. Then 2 or more different normal velocities yield the true full velocity (in the least squares sense).
- We could assume that velocity varies smoothly everywhere [Horn and Schunck AI1981] and regularize a smoothness term across the image.

2D Lucas and Kanade 1981

- Given I_x, I_y and I_t at a single pixel, we can compute $\vec{v}_n$ as $\vec{v}_n = \vec{v} \cdot \hat{n}$, where $v_n = \frac{-I_t}{||\nabla I||_2}$ and $\hat{n} = \frac{\nabla I}{||\nabla I||_2}$ are as before.
- Given an $k = n \times n$ neighbourhood with the same velocity $\vec{v}$ we can write

$$\underbrace{\begin{bmatrix} n_{x1} & n_{y1} \\ n_{x2} & n_{y2} \\ \vdots & \vdots \\ n_{xk} & n_{yk} \end{bmatrix}}_{N} \underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\vec{v}} = \underbrace{\begin{bmatrix} v_{n1} \\ v_{n2} \\ \vdots \\ v_{nk} \end{bmatrix}}_{B}$$

  which we can write as $\underbrace{N}_{k\times 2}\underbrace{\vec{v}}_{2\times 1} = \underbrace{B}_{k\times 1}$.

- For $k \geq 2$ we can solve this system as $\vec{v} = (N^T N)^{-1} N^T B$.

2D Horn and Schunck 1981

- Horn and Schunck combined the motion constraint equation with a global smoothness term to constrain the estimated velocity field $\vec{v} = (u, v)$, minimizing:

$$f = \int_D (\nabla I \cdot \vec{v} + I_t)^2 + \lambda^2 \left(u_x^2 + u_y^2 + v_x^2 + v_y^2\right) dxdy$$

  defined over a domain D (the image), where the magnitude of λ reflects the relative influence of the smoothness term.

- We use the Euler-Lagrange equations:

$$\begin{aligned} f_u - \frac{df_{u_x}}{dx} - \frac{df_{u_y}}{dy} &= 0 \\ f_v - \frac{df_{v_x}}{dx} - \frac{df_{v_y}}{dy} &= 0, \end{aligned}$$

where:

$$
\begin{aligned}
f_u &= 2I_x(I_x u + I_y v + I_t) \\
f_v &= 2I_y(I_x u + I_y v + I_t) \\
f_{u_x} &= 2\alpha^2 u_x \\
f_{u_y} &= 2\alpha^2 u_y \\
f_{v_x} &= 2\alpha^2 v_x \\
f_{v_y} &= 2\alpha^2 v_y \\
\frac{df_{u_x}}{dx} &= 2\alpha^2 u_{xx} \\
\frac{df_{u_y}}{dy} &= 2\alpha^2 u_{yy} \\
\frac{df_{v_x}}{dx} &= 2\alpha^2 v_{xx} \\
\frac{df_{v_y}}{dy} &= 2\alpha^2 v_{yy}.
\end{aligned}
$$

- Since $\nabla^2 u = u_{xx} + u_{yy}$ and $\nabla^2 v = v_{xx} + v_{yy}$ we can rewrite the Euler-Lagrange equations as:

$$\begin{aligned} I_x^2 u + I_x I_y v + I_x I_t &= \alpha^2 \nabla^2 u \\ I_x I_y u + I_y^2 v + I_y I_t &= \alpha^2 \nabla^2 v. \end{aligned}$$

  Using $\nabla^2 u \approx \bar{u} - u$ and $\nabla^2 v \approx \bar{v} - v$ we get

$$\begin{aligned} (\alpha^2 + I_x^2) u + I_x I_y v &= (\alpha^2 \bar{u} - I_x I_t) \\ I_x I_y u + (\alpha^2 + I_y^2) v &= (\alpha^2 \bar{v} - I_y I_t), \end{aligned}$$

- We can solve for u and v as:

$$\begin{aligned} (\alpha^2 + I_x^2 + I_y^2) u &= (\alpha^2 + I_y^2)\bar{u} - I_x I_y \bar{v} - I_x I_t \\ (\alpha^2 + I_x^2 + I_y^2) v &= -I_x I_y \bar{u} + (\alpha^2 + I_y^2)\bar{v} - I_y I_t, \end{aligned}$$

which can be written as:

$$\begin{aligned}(\alpha^2 + I_x^2 + I_y^2)(u - \bar{u}) &= -I_x \left[I_x\bar{u} + I_y\bar{v} + I_t\right] \\ (\alpha^2 + I_x^2 + I_y^2)(v - \bar{v}) &= -I_y \left[I_x\bar{u} + I_y\bar{v} + I_t\right].\end{aligned}$$

- Gauss-Seidel iterative equations that minimize these equations are:

$$\begin{aligned}u^{k+1} &= \bar{u}^k - \frac{I_x[I_x\bar{u}^k + I_y\bar{v}^k + I_t]}{\alpha^2 + I_x^2 + I_y^2} \text{ and} \\ v^{k+1} &= \bar{v}^k - \frac{I_y[I_x\bar{u}^k + I_y\bar{v}^k + I_t]}{\alpha^2 + I_x^2 + I_y^2}.\end{aligned}$$

- Here k denotes the iteration number, v_y^0 and v_y^0 denote initial velocity estimates which are typically set to zero and $\bar{u}^k$ and $\bar{v}^k$ denote neighbourhood averages of u^k and v^k. Iterations are stopped if k reaches a preset value, i.e. 100, or if the norm of the velocity field differences at iterations k and $k+1$ is less than some preset threshold.

Intensity Differentiation

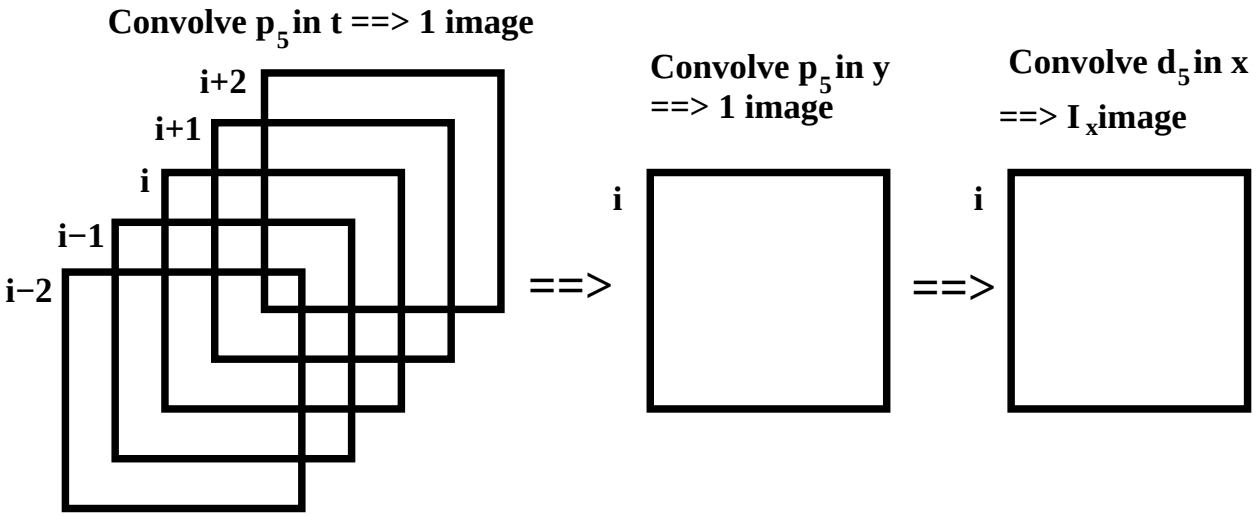
- One way to compute intensity derivatives is via convolution of Simoncelli's Matched/balanced filters [ICIP1994] compute I_x, I_y and I_t.
- The filter coefficients are:

n	p_5	d_5
0	0.036	-0.108
1	0.249	-0.283
2	0.431	0.0
3	0.249	0.283
4	0.036	0.108

Table 1: Simoncelli's 5-point Matched/Balanced Kernels

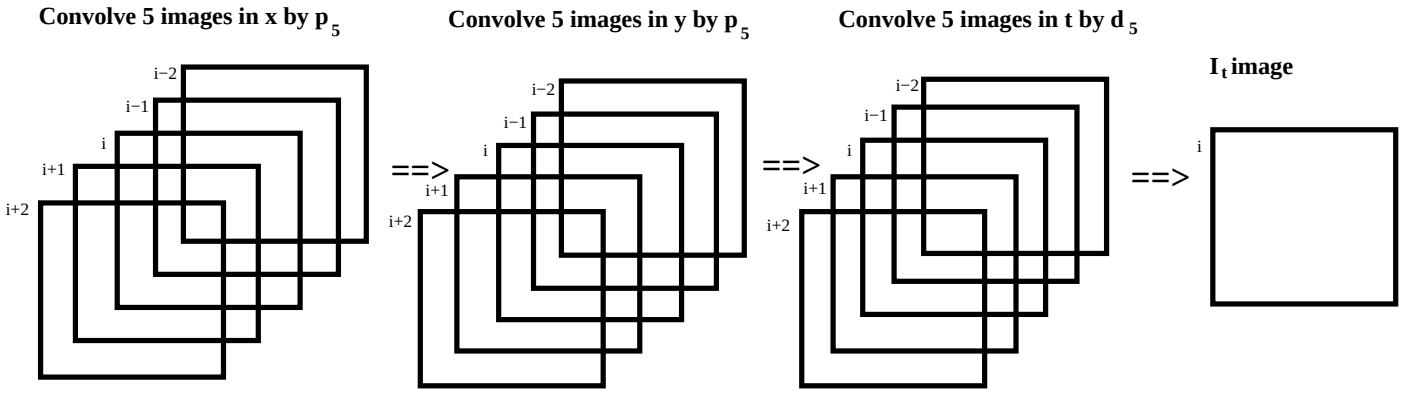
Convolution Steps for Computing I_x

- To compute I_x in 2D for frame i is some sequence, we first convolve the smoothing kernel, p_5, in the t dimension to reduce the 5 images to 1 image, then convolve the smoothing kernel p_5 on that result in the y dimension and finally convolve the differentiation kernel, d_5, on that 2^{nd} result in the x dimension to obtain I_x. I_y is computed in a similar way.



Convolution Steps for Computing I_t

- To compute I_t in 2D for frame i in some sequence, we first convolve p_5 in the x dimension and then on that result in the y dimension for each of frames $i-2$, $i-1$, i, $i+1$ and $i+2$. This yields 5 smoothed images in x and y. We then differentiate these images in the t dimension using d_5 to get I_t at frame i.



3D Optical Flow

- Usually, by 3D optical flow we mean 3D volumetric flow: at voxel, (x, y, z), what is the 3D velocity (U, V, W)? An example of volumetric data: 20 volumes of gated MRI data of 1 beat of a human heart. The 3D motion constraint equations uses intensity derivatives in x, y and z, as well as t.
- 3D optical flow on a moving surface (such as a growing leaf) is called **range** flow and also produces 3D velocity (U, V, W) on all surface points of some scanned object that is moving over time. Now the 3D constraint equation uses x and y derivatives of Z, the depth value (either scanned or computed) at each surface point. Range flow is also often called **scene** flow.

The **3D Intensity Motion Constraint Equation** is a simple extension of the 2D motion constraint equation. Consider a small 3D $n \times n \times n$ block at (x, y, z) at time t moving to $(x + \delta x, y + \delta y, z + \delta z)$ at time $t + \delta t$.

Figure 6: A small $n \times n \times n$ 3D neighbourhood of voxels centered at (x, y, z) at time t moving to $(x + \delta x, y + \delta y, z + \delta z)$ at time $t + \delta t$.

3D Intensity Motion Constraint Equation and the 3D Aperture Problem

- We assume a 3D voxel $I(x, y, z)$ at time t moves with a displacement $(\delta x, \delta y, \delta z)$ over time δt. Since $I(x, y, z, t)$ and $I(x + \delta x, y + \delta y, z + \delta z, t + \delta t)$ are the same we can perform a 1^{st} order Taylor series expansion and obtain (as in the 2D case):

$$I_x U + I_y V + I_z W = \nabla I \cdot \vec{V} = -I_t,$$

  I_x, I_y, I_z and I_t are 3D spatio-temporal intensity derivatives computed via Simoncelli convolution.

- The 3D velocity is $\vec{V} = (U, V, W)$
- This equation describes a plane in 3D space. Any point on that plane is possibly the correct 3D velocity. The velocity on the plane that is

closest to the origin is called the **plane normal** velocity (the velocity normal to a local intensity planar structure)

- The **line normal** velocity is the velocity on the line caused by the intersection of 2 planes that is closest to the origin. of course, is three planes intersect at a single point, that point is the full 3D velocity.

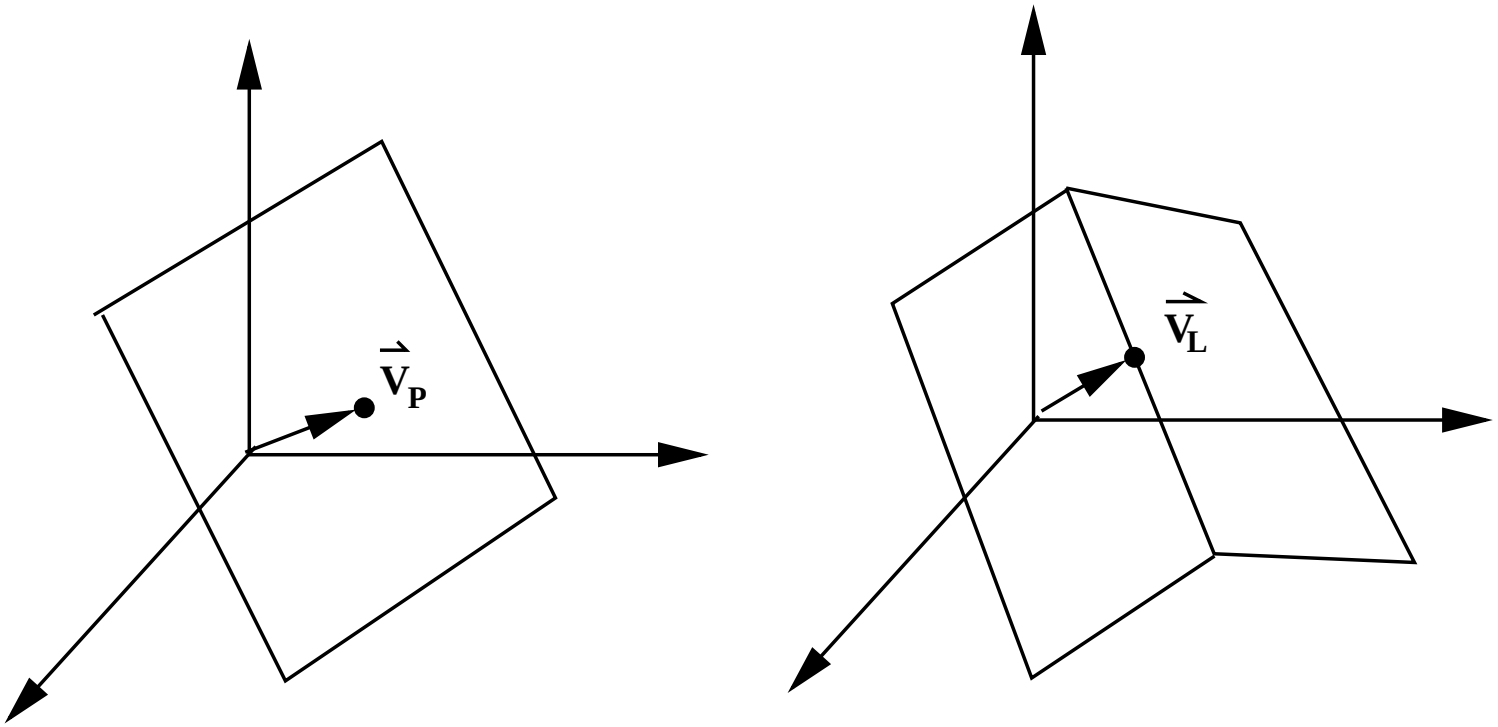


Figure 7: Graphical illustrations of the 3D plane and line normal velocities.

3D Lucas and Kanade

- From the 3D the motion constraint equation we have:

$$I_x U + I_y V + I_z W = -I_t,$$

  where I_x, I_y, I_z and I_t are the 3D intensity derivatives in a $n \times n \times n$ neighbourhood centered at voxel (x, y, z) and $\vec{V} = (U, V, W)$ is that neighbourhood's (assumed) constant 3D velocity.

- Given I_x, I_y, I_z and I_t at a single pixel, we can compute $\vec{V}_n$ as $\vec{V}_n = \vec{V} \cdot \hat{n}$, where $V_n = \frac{-I_t}{||\nabla I||_2}$, $\hat{n} = \frac{\nabla I}{||\nabla I||_2}$ and $\nabla I = (I_x, I_y, I_z)$.

- Given an $k = n \times n \times n$ neighbourhood with the same velocity $\vec{V}$ we

can write

$$\underbrace{\begin{bmatrix} n_{x1} & n_{y1} & n_{z1} \\ n_{x2} & n_{y2} & n_{z2} \\ \vdots & \vdots & \vdots \\ n_{xk} & n_{yk} & n_{zk} \end{bmatrix}}_{N} \underbrace{\begin{bmatrix} U \\ V \\ W \end{bmatrix}}_{\vec{V}} = \underbrace{\begin{bmatrix} V_{n1} \\ V_{n2} \\ \vdots \\ V_{nk} \end{bmatrix}}_{B}$$

which we can write as $\underbrace{N}_{k\times 3}\underbrace{\vec{V}}_{3\times 1} = \underbrace{B}_{k\times 1}$.

- For $k \geq 3$ we can solve this system as $\vec{V} = (N^T N)^{-1} N^T B$.

3D Horn and Schunck

- 3D Horn and Schunck regularization becomes:

$$F = \int_D (I_x U + I_y V + I_z W + I_t)^2 + \alpha^2 \left(U_x^2 + U_y^2 + U_z^2 + V_x^2 + V_y^2 + V_z^2 + W_x^2 + W_y^2 + W_z^2 \right).$$

- We use the Euler-Lagrange equations:

$$\begin{aligned} F_U - \frac{d}{dX} F_{U_X} - \frac{d}{dY} F_{U_Y} - \frac{d}{dZ} F_{U_Z} - \frac{d}{dt} F_{U_t} &= 0, \\ F_V - \frac{d}{dX} F_{V_X} - \frac{d}{dY} F_{V_Y} - \frac{d}{dZ} F_{V_Z} - \frac{d}{dt} F_{V_t} &= 0, \\ F_W - \frac{d}{dX} F_{W_X} - \frac{d}{dY} F_{W_Y} - \frac{d}{dZ} F_{W_Z} - \frac{d}{dt} F_{W_t} &= 0. \end{aligned}$$

where:

$$
\begin{aligned}
F_U &= 2Z_X(Z_X U + Z_Y V + Z_Z W + Z_t) \\
F_V &= 2Z_Y(Z_X U + Z_Y V + Z_Z W + Z_t) \\
F_W &= 2Z_Z(Z_X U + Z_Y V + Z_Z W + Z_t) \\
F_{U_X} &= 2\alpha^2 U_X \\
F_{U_Y} &= 2\alpha^2 U_Y \\
F_{U_Z} &= 2\alpha^2 U_Z \\
F_{U_t} &= 2\alpha^2 U_t \\
F_{V_X} &= 2\alpha^2 V_X \\
F_{V_Y} &= 2\alpha^2 V_Y \\
F_{V_Z} &= 2\alpha^2 V_Z \\
F_{V_t} &= 2\alpha^2 V_t \\
F_{W_X} &= 2\alpha^2 W_X \\
F_{W_Y} &= 2\alpha^2 W_Y \\
F_{W_Z} &= 2\alpha^2 W_Z \\
F_{W_t} &= 2\alpha^2 W_t
\end{aligned}
$$

and

$$\begin{aligned}
\frac{dF_{U_X}}{dX} &= 2\alpha^2 U_{XX}, & \frac{dF_{V_Z}}{dZ} &= 2\alpha^2 V_{ZZ} \\
\frac{dF_{U_Y}}{dY} &= 2\alpha^2 U_{YY}, & \frac{dF_{V_t}}{dt} &= 2\alpha^2 V_{tt} \\
\frac{dF_{U_Z}}{dZ} &= 2\alpha^2 U_{ZZ}, & \frac{dF_{W_X}}{dX} &= 2\alpha^2 W_{XX} \\
\frac{dF_{U_t}}{dt} &= 2\alpha^2 U_{tt}, & \frac{dF_{W_Y}}{dY} &= 2\alpha^2 W_{YY} \\
\frac{dF_{V_X}}{dX} &= 2\alpha^2 V_{XX}, & \frac{dF_{W_Z}}{dZ} &= 2\alpha^2 W_{ZZ} \\
\frac{dF_{V_Y}}{dY} &= 2\alpha^2 V_{YY}, & \frac{dF_{W_t}}{dt} &= 2\alpha^2 W_{tt}.
\end{aligned}$$

- Since $\nabla^2 U = U_{XX} + U_{YY} + U_{ZZ} + U_{tt}$, $\nabla^2 V = V_{XX} + V_{YY} + V_{ZZ} + V_{tt}$ and $\nabla^2 W = W_{XX} + W_{YY} + W_{ZZ} + W_{tt}$ we can rewrite the

Euler-Lagrange equations as:

$$\begin{aligned} Z_X^2 U + Z_X Z_Y V + Z_X Z_Z W + Z_X Z_t &= \alpha^2 \nabla^2 U \\ Z_X Z_Y U + Z_Y^2 V + Z_Y Z_Z W + Z_Y Z_t &= \alpha^2 \nabla^2 V \\ Z_X Z_Z U + Z_Y Z_Z V + Z_Z^2 W + Z_Z Z_t &= \alpha^2 \nabla^2 W \end{aligned}$$

- Using $\nabla^2 U \approx \bar{U} - U$, $\nabla^2 V \approx \bar{V} - V$ and $\nabla^2 W \approx \bar{W} - W$ we can write:

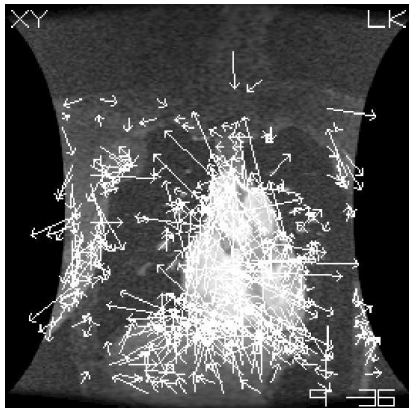
$$\begin{aligned} (\alpha^2 + Z_X^2) U + Z_X Z_Y V + Z_X Z_Z W &= (\alpha^2 \bar{U} + Z_X Z_t) \\ Z_X Z_Y U + (\alpha^2 + Z_Y^2) V + Z_Y Z_Z W &= (\alpha^2 \bar{V} + Z_Y Z_t) \\ Z_X Z_Z U + Z_Y Z_Z V + (\alpha^2 + Z_Z^2) W &= (\alpha^2 \bar{W} + Z_Z Z_t). \end{aligned}$$

- The Gauss-Seidel iterative equations can be written as:

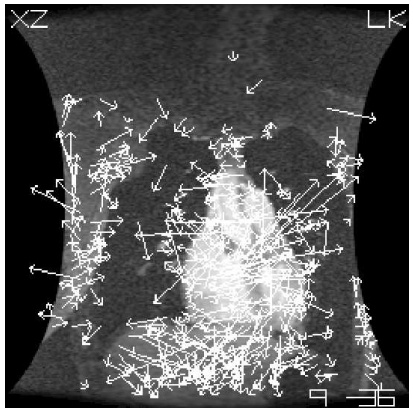
$$
\begin{aligned}
U^{k+1} &= \bar{U}^k - \frac{I_x \left[I_x \bar{U}^k + I_y \bar{V}^k + I_z \bar{W}^k + I_t \right]}{(\alpha^2 + I_x^2 + I_y^2 + I_z^2)}, \\
V^{k+1} &= \bar{V}^k - \frac{I_y \left[I_x \bar{U}^k + I_y \bar{V}^k + I_z \bar{W}^k + I_t \right]}{(\alpha^2 + I_x^2 + I_y^2 + I_z^2)} \text{ and} \\
W^{k+1} &= \bar{W}^k - \frac{I_z \left[I_x \bar{U}^k + I_y \bar{V}^k + I_z \bar{W}^k + I_t \right]}{(\alpha^2 + I_x^2 + I_y^2 + I_z^2)}.
\end{aligned}
$$

- $\bar{U}^k$, $\bar{V}^k$ and $\bar{W}^k$ are $n \times n \times n$ averages of neighbourhoods of velocities at iteration k. $\bar{U}^0$, $\bar{V}^0$ and $\bar{W}^0$ are typically set to 0.0.

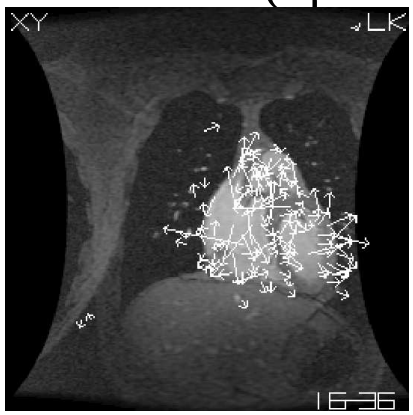
3D L&K OF for Gated MRI Cardiac Data



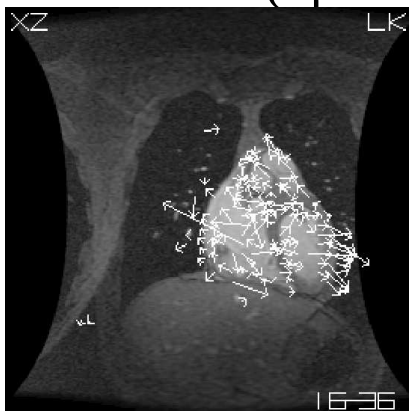
LK-XY-9-36 (5phase)



LK-XZ-9-36 (5phase)



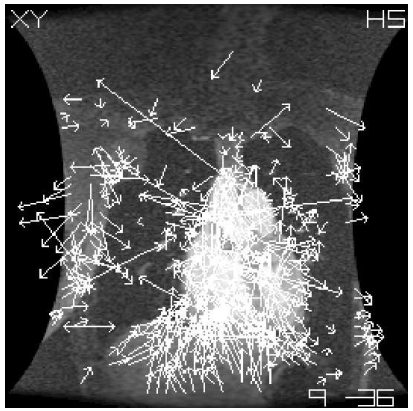
LK-XY-16-36 (10phase)



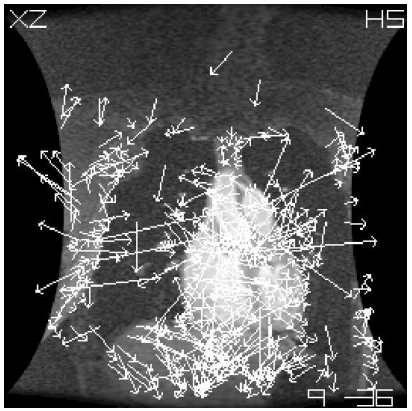
LK-XZ-16-36 (10phase)

Figure 8: The Lucas and Kanade XY and XZ flow fields superimposed on the 36^{th} slice of the 9^{th} and 16^{th} volumes of the 5phase and 10phase datasets. The eigenvalue threshold λ_1 was 1.0.

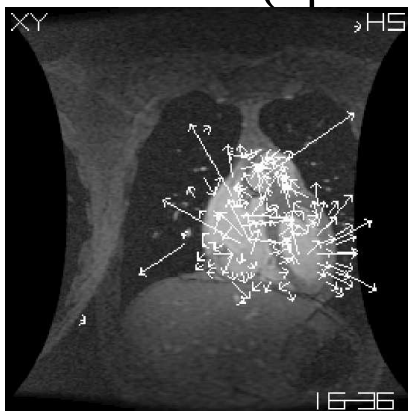
3D H&S OF for Gated MRI Cardiac Data



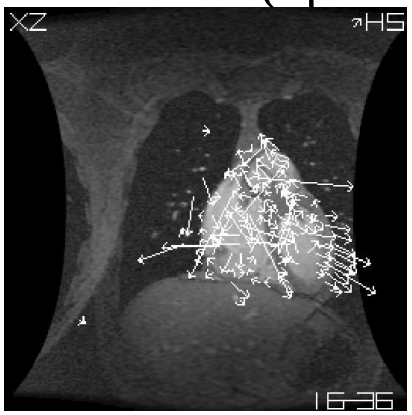
HS-XY-9-36 (5phase)



HS-XZ-9-36 (5phase)



HS-XY-16-36 (10phase)



HS-XZ-16-36 (10phase)

Figure 9: The Horn and Schunck XY and XZ flow fields superimposed on the 36^{th} slice of the 9^{th} and 16^{th} volumes of the 5phase and 10phase datasets for 100 iterations. $\alpha = 1.0$.

3D Range Motion Constraint Equation

- We can compute 3D Range Flow (3D Optical Flow on a 3D Surface) from 3D depth data measured by a ShapeGrabber range scanner on rigid and non-rigid surfaces [CVUI2002]. One example is the range flow for a plant leaf (the surface can be deformable) [ECCV2000].
- The 3D Range Constraint Equation is $Z_x U + Z_y V + W - 1 = 0$. Here Z_x and Z_y are depth derivatives and not intensity derivatives.
- Lucas and Kanade and Horn and Schunck like 3D surface optical flow can be computed [DAGM1999].
- Range flow (from time-varying depth data) can be fused with 2D optical flow (from time-varying intensity data) to produce full 3D velocity on moving flat surfaces, where neither range or optical flow by themselves, can do this [ICIP2000].

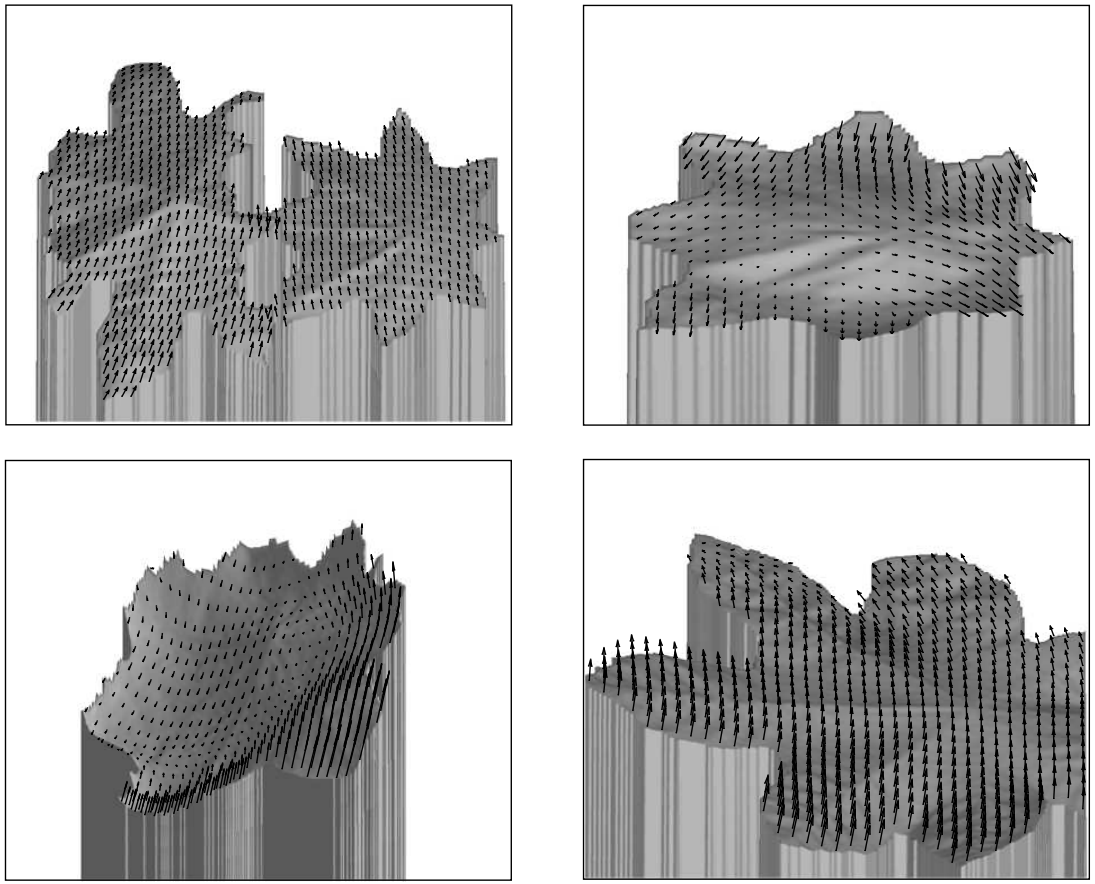


Figure 10: Example 3D range flow for Castor oil plant leaves.

3D Scene Flow

- A typical approach: In a stereo image sequence, compute the time-varying depth maps using a stereo algorithm,
- Compute range or scene flow from these time-varying depth maps.
- One point of view: optical flow is not practical on real imaginary but stereo is much better: so use stereo to get depth maps and then compute range flow on these depth maps and their derivative values. One example of such work: “Efficient Dense Scene Flow from Sparse or Dense Stereo Data”, Andreas Wedel, Clemens Rabe, Tobi Vaudrey, Thomas Brox, Uwe Franke and Daniel Cremers, ECCV, October, 2008.